Unit 7

Equation of state: Degenerate gas

7.1 Completely degenerate gas

- The ideal gas law (because of Maxwell-Boltzmann statistics) breaks down at sufficiently high densities and/or low temperatures.
- Consider the extreme case where $T \to 0$ at fixed density.
- From Equation (6.7), the Maxwell distribution peaks at zero momentum where all the particles want to pile up. They want to be in the lowest energy state, which is zero.
- There's a limit to how close fermions can come, based on the Pauli exclusion principle.
- So instead, we must use Fermi-Dirac statistics and not Maxwell-Boltzmann.
- Consider first the most interesting terms in Equation (6.1)

$$f(p) = \frac{1}{\mathrm{e}^{(E(p)-\mu_c)/kT} + 1}, \tag{7.1}$$

  where we've taken a reference energy level $E_j = 0$.
- As $T \to 0$, f goes to 1 or 0 depending on the sign of $E - \mu_c$.
- The function is discontinuous at the Fermi momentum p_F, or at energy E_F.
- For fermions such as electrons, with spin 1/2, the degeneracy factor in Equation (6.1) $g = 2$.
- The chemical potential is the Fermi energy $\mu_c = E_F$, up to which all the quantum states are filled. This is what is meant by "degeneracy."
- It is convenient to introduce the dimensionless momentum $x = p/mc$ and Fermi momentum $x_F = p_F/mc$.
- The integration to obtain the number density of electrons is therefore

$$n_e = \frac{8\pi}{h^3}\int_0^{p_F} p^2\,\mathrm{d}p = 8\pi\left(\frac{h}{m_e c}\right)^{-3}\int_0^{x_F} x^2\,\mathrm{d}x = \frac{8\pi}{3}\left(\frac{h}{m_e c}\right)^{-3} x_F^3 = 5.865\times 10^{29}\,x_F^3\ \mathrm{cm}^{-3}. \tag{7.2}$$

  Note that $p_F \sim n_e^{1/3}$.
- It is common, yet confusing, to remove the subscript for Fermi and just to write

$$n_e = 5.865\times 10^{29}\,x^3\ \mathrm{cm}^{-3}. \tag{7.3}$$

- If we reintroduce the electron mean molecular weight, as in Equation (5.14), we can get this in terms of mass density

$$\frac{\rho}{\mu_e} = \frac{8\pi m_{\mathrm{u}}}{3}\left(\frac{h}{m_e c}\right)^{-3} x_{\mathrm{F}}^3 = 9.74 \times 10^5\, x_{\mathrm{F}}^3 \ \mathrm{g\,cm^{-3}}. \tag{7.4}$$

This is interesting in that it is a way to determine the Fermi momentum or energy if provided a value for ρ/μ_e. Also note the densities of about 10^6, which are typical of white dwarfs.

- The electron pressure (the pressure due to degenerate electrons, not ions) from Equation (6.5) is therefore

$$P_e = \frac{8\pi}{3}\frac{m_e^4 c^5}{h^3}\int_0^{x_{\mathrm{F}}} \frac{x^4}{(1+x^2)^{1/2}}\mathrm{d}x = Cf(x), \tag{7.5}$$

where $C = \pi m_e^4 c^5/3h^3 = 6.002 \times 10^{22}\,\mathrm{dyne\,cm^{-2}}$.

- The function $f(x)$ is

$$f(x) = x(2x^2-3)(1+x^2)^{1/2} + 3\sinh^{-1} x. \tag{7.6}$$

- Similarly, the internal energy density from Equation (6.6) is

$$u_e = 8\pi\frac{m_e^4 c^5}{h^3}\int_0^{x_{\mathrm{F}}} x^2\left[(1+x^2)^{1/2}-1\right]\mathrm{d}x = Cg(x), \tag{7.7}$$

where C is the same as before.

- The function $g(x)$ is

$$g(x) = 8x^3\left[(1+x^2)^{1/2}-1\right] - f(x). \tag{7.8}$$

- These are very general expressions in terms of x for the pressure and energy density.

IN CLASS WORK

Show that x discriminates between the nonrelativistic regime $x \ll 1$ and the relativistic regime $x \gg 1$. Use Equation (6.3) and Equation (6.4).

Answer: Since $x = p/m_e c$, we need to know what the momentum is. It's not simply $p = mv$. We can compute the velocity and then the momentum.

$$v = \frac{\partial E}{\partial p} = \frac{p/m}{\left(1+\frac{p^2}{m^2c^2}\right)^{1/2}}$$

Solving for p gives

$$p = \frac{mv}{\sqrt{1-v^2/c^2}},$$

which is expected. Therefore,

$$x = \frac{v/c}{\sqrt{1-v^2/c^2}}.$$

Another useful form is to solve for the velocity ratio in terms of x:

$$\frac{v^2}{c^2} = \frac{x^2}{1+x^2}.$$

In any case, for electron velocities near the speed of light, x becomes very large.

- Let us first consider the case for **nonrelativistic electrons** where $x \ll 1$. In this limit, to first order

$$\begin{aligned} f(x) &\approx \frac{8}{5}x^5, \\ g(x) &\approx \frac{12}{5}x^5. \end{aligned}$$

- Using Equation (7.5) we thus have

$$P_e = \frac{8\pi m_e^4 c^5}{15h^3}x^5. \tag{7.9}$$

- Relating this through the density in Equation (7.2) to remove x, and then using the electron mean molecular weight in Equation (5.14), we arrive at the final expression

$$P_\mathrm{e} = 1.0036 \times 10^{13} \left(\frac{\rho}{\mu_\mathrm{e}}\right)^{5/3} \mathrm{dyne\,cm^{-2}}. \tag{7.10}$$

  This is the equation of state for a fully degenerate nonrelativistic electron gas.

- Carrying out the same exercise for the internal energy, we find

$$u_\mathrm{e} = \frac{3}{2}P_\mathrm{e}. \tag{7.11}$$

- Thus, the equation of state for this nonrelativistic gas has characteristics of an ideal monatomic gas (see Equation (6.16)).

- Let us now consider the case for **relativistic electrons** where $x \gg 1$. In this limit, to first order

$$\begin{aligned} f(x) &\approx 2x^4, \\ g(x) &\approx 6x^4. \end{aligned}$$

- Now, the pressure is

$$P_\mathrm{e} = \frac{2\pi m_e^4 c^5}{3h^3}x^4, \tag{7.12}$$

  which after plugging in constants and introducing n_e and ρ gives

$$P_\mathrm{e} = 1.243 \times 10^{15} \left(\frac{\rho}{\mu_\mathrm{e}}\right)^{4/3} \mathrm{dyne\,cm^{-2}}. \tag{7.13}$$

- Similarly for the energy we have

$$u_\mathrm{e} = 3P_\mathrm{e}. \tag{7.14}$$

- The transition from non- to relativistic states is smooth in x. Note the exponents on the density, which we will come back to these later (polytropes).

PROBLEM 7.1: [10 pts]: Find the ratio of the electron degeneracy pressure to the electron ideal gas pressure in the center of the (current) Sun (assuming the center could be degenerate in electrons). Let $T = 15 \times 10^6$ K, $\rho = 150\,\mathrm{g\,cm^{-3}}$, and abundances $X = 0.35$, $Z = 0.02$. Prove that you are using the correct expression for the electron degeneracy pressure.

7.2 Partially degenerate gas

- The previous section considered an ideal zero-temperature gas. But if the temperature is finite, then the Fermi-Dirac function is not a simple step function and needs to be evaluated numerically.
- When this is done, the temperature dependence of the equation of state is realized.
- The typical expressions are just expansions of the Fermi function in powers of T.
- Qualitatively though, as temperature is increased some amount, only the electrons near the Fermi energy have the freedom to move to higher states and smear out the step function. Only if a temperature equivalent to about the Fermi energy $E_{\mathrm{F}} = k_{\mathrm{B}}T$ is achieved can particles deep within the Fermi sea find unoccupied levels at higher energies. If that's the case, the gas becomes more like a classical one.
- So the transition from degeneracy to non degeneracy can roughly be considered to occur when the temperature of the gas is near the Fermi energy.
- We will not spend more time on this right now, but keep in mind that these states of matter are not typically homogeneous, but rather a mixture.