Unit 5

Mean molecular weight

- Another concept before we dive into equations of state is the mean molecular weight μ, which is important to understand.

- Stellar interiors have a mixture of atoms of different elements and various ionizations.

- Consider the mean mass $\overline{m}$ per particle

$$\overline{m} = \frac{\sum_j n_{j,\mathrm{I}} m_{j,\mathrm{I}} + n_\mathrm{e} m_\mathrm{e}}{\sum_j n_{j,\mathrm{I}} + n_\mathrm{e}} \approx \frac{\sum_j n_{j,\mathrm{I}} m_{j,\mathrm{I}}}{\sum_j n_{j,\mathrm{I}} + n_\mathrm{e}}, \tag{5.1}$$

  where $n_{j,\mathrm{I}}$ is the ion number density of ion j, $m_{j,\mathrm{I}}$ is its mass, and n_e and m_e are the numbers and mass of the electron (and then we ignore the electron mass).

- The mass of the jth ion is approximately its number of protons and neutrons (A_j) times the amu, or $m_{j,\mathrm{I}} = A_j m_\mathrm{u}$.

- So then we define

$$\mu = \frac{\overline{m}}{m_\mathrm{u}} = \frac{\sum_j n_{j,\mathrm{I}} A_j}{\sum_j n_{j,\mathrm{I}} + n_\mathrm{e}}. \tag{5.2}$$

  This can be interpreted as the average mass per particle (ion, electron, etc.) in units of the amu.

- Note that the total particle number density in the gas is

$$n = n_e + n_\mathrm{I} = n_e + \sum_j n_{j,\mathrm{I}} = \sum_j (1 + Z_j) n_{j,\mathrm{I}}, \tag{5.3}$$

  since one ionized atom contributes 1 nucleus plus Z_j electrons. The total $n_\mathrm{e} = \sum_j n_{j,\mathrm{I}} Z_j$, where Z_j is the "charge" of each nucleus.

- In general though, the electron density (or level of ionization) is complicated and derived from the Saha equation. Such an equation gives ionization fractions y_i such that the electron number density would be $n_\mathrm{e} = \sum_j n_{j,\mathrm{I}} y_j Z_j$ (see later).

- But to be more useful, it's easier to express the number densities in terms of mass fractions x_i, where $\sum_i x_i = 1$.

- The number densities we looked at earlier are for some species i are

$$n_i = \frac{\rho}{m_\mathrm{u}} \frac{x_i}{A_i}. \tag{5.4}$$

  Think of this as the mass per unit volume of species i (ρx_i), in units of 1 ion of species i ($m_\mathrm{u} A_i$).

IN CLASS WORK

Imagine a star where 92% of all particles are hydrogen nuclei and 8% of them are helium nuclei. What are the mass fractions of hydrogen and helium?

$$\begin{aligned}
92 &= \frac{\rho x_{\rm H}}{m_{\rm u}} \\
8 &= \frac{\rho x_{\rm He}}{4m_{\rm u}} \\
x_{\rm H} &= 92m_{\rm u}/\rho \\
x_{\rm He} &= 32m_{\rm u}/\rho \\
92m_{\rm u}/\rho + 32m_{\rm u}/\rho &= 124m_{\rm u}/\rho = 1 \\
m_{\rm u}/\rho &= 1/124 \\
x_{\rm H} &= 92/124 = 0.7419 \\
x_{\rm He} &= 32/124 = 0.2581.
\end{aligned}$$

- So using this, we now have

$$\mu = \frac{\sum_i \frac{\rho}{m_{\rm u}} x_i}{\sum_i \frac{\rho x_i}{m_{\rm u} A_i} + n_e}, \tag{5.5}$$

or

$$\mu = \frac{\sum_i \frac{\rho}{m_{\rm u}} x_i}{\sum_i \frac{\rho x_i}{m_{\rm u} A_i}(1+Z_i)}. \tag{5.6}$$

- One cleaner way of writing this is

$$\mu^{-1} = \frac{\sum_i x_i/A_i(1+Z_i)}{\sum_i x_i} = \sum_i \frac{x_i}{A_i}(1+Z_i). \tag{5.7}$$

- For example, for a neutral gas ($Z = 0$) we have

$$\mu^{-1} = \sum_i \frac{x_i}{A_i} \approx \left(X + \frac{Y}{4} + \frac{Z}{\overline{A}_i}\right)^{-1}, \tag{5.8}$$

where it is standard to write mass fractions X for hydrogen, Y for helium, and Z for everything else (metals), where $X + Y + Z = 1$.

- $\overline{A}_i$ is an average over metals, which at solar composition is about 15.5.

- For a fully ionized gas

$$\mu^{-1} \approx \sum_i \frac{x_i}{A_i}(1+Z_i) \approx 2X + \frac{3}{4}Y + \frac{1}{2}Z, \tag{5.9}$$

or

$$\mu \approx \frac{4}{3+5X-Z}, \tag{5.10}$$

where for metals we usually approximate $(1+Z_i)/A_i \approx 1/2$ (roughly equal number of protons and neutrons). We've eliminated Y in this expression through $Y = 1 - X - Z$.

IN CLASS WORK

Compute the mean molecular weight for (1) the ionized solar photosphere, where we have 90% hydrogen, 9% helium, and 1% heavy elements by mass, (2) the ionized solar interior where 71% hydrogen, 27% helium, and 2% heavy elements by mass, (3) completely ionized hydrogen, (4) completely ionized helium, and finally (5) neutral gas at the solar interior abundance.

Answer: (1) For the photosphere we can write

$$\mu^{-1} = 0.9\frac{2}{1} + 0.09\frac{3}{4} + 0.01\frac{1}{2} = 1.8725,$$

or $\mu \approx 0.53$.
(2) For the interior we can write

$$\mu^{-1} = 0.71\frac{2}{1} + 0.27\frac{3}{4} + 0.02\frac{1}{2} = 1.63,$$

or $\mu \approx 0.61$.
(3) For hydrogen, we will take $X = Z = A = 1$, and find then that $1/\mu = 2$.
(4) For helium, $X = Z = 0$ and $Y = 1$, so $\mu = 4/3$.
(5) For a neutral gas, we have

$$\mu^{-1} = 0.71 + 0.27\frac{1}{4} + 0.02\frac{1}{15.5} = 0.779,$$

or $\mu \approx 1.28$.

- From the above, we can also consider separately the mean molecular weight for ions and electrons.
- For ions, define $\mu_{\rm I}$ as
$$n_{\rm I} = \frac{\rho}{\mu_{\rm I} m_{\rm u}}. \tag{5.11}$$
Recall that
$$n_{\rm I} = \sum_j n_{j,{\rm I}} = \frac{\rho}{m_{\rm u}} \sum_j \frac{x_j}{A_j}. \tag{5.12}$$
So that
$$\mu_{\rm I} = \left(\sum_j \frac{x_j}{A_j}\right)^{-1}. \tag{5.13}$$
- This result should make sense, since above in Equation (5.8) we did not consider electrons.
- For electrons it's a bit harder since not all electrons need be free. But we will still define the *mean molecular weight per election* $\mu_{\rm e}$:
$$n_{\rm e} = \frac{\rho}{\mu_{\rm e} m_{\rm u}} \tag{5.14}$$
- Fully ionized, each atom contributes Z electrons. If an ion is partially ionized, we can consider the fraction yZ. (To compute the proper fraction of ionization of a gas (n_e), one needs to use the *Saha equation*).
- As before then
$$n_e = \sum_j n_{e,j} = \sum_j n_{j,{\rm I}} y_j Z_j = \frac{\rho}{m_{\rm u}} \sum_j \left(\frac{x_j}{A_j}\right) y_j Z_j, \tag{5.15}$$

which defines

$$\mu_e = \left(\sum_j \frac{x_j y_j Z_j}{A_j} \right)^{-1}. \tag{5.16}$$

- So finally

$$n = n_e + n_\mathrm{I} = \frac{\rho}{\mu m_\mathrm{u}}, \tag{5.17}$$

where

$$\mu = \left(\frac{1}{\mu_\mathrm{I}} + \frac{1}{\mu_e} \right)^{-1}. \tag{5.18}$$

IN CLASS WORK

Compute an expression for μ_e in the deep stellar interior as a function only of X. Ignore metals.

Answer: Fully ionized case. We can write

$$\begin{aligned} \mu_\mathrm{e} &\approx \left(\frac{1}{1} X + \frac{2}{4} Y \right)^{-1} \\ &= \left(X + \frac{1}{2}(1-X) \right)^{-1} \\ &= \left(\frac{X+1}{2} \right)^{-1} = \frac{2}{1+X}. \end{aligned}$$

This should make sense. For a full H gas, the mean mass of particles per number of electrons (1/1) is 1. For a He gas ($X = 0$), we have a mass of 4 divided by 2 electrons, or $\mu_e = 2$.