Part I

Stellar Structure

Unit 1

Equilibrium and time scales

1.1 Energy equilibrium

- Consider a thin layer at location r and of width $\mathrm{d}r$ in the nuclear burning region of a star. For energy equilibrium, the net flow of energy crossing this region's boundaries should be equal to the energy generation in the region.
- Let ε be the energy generated per gram of material per second, so the energy generated per second in $(r,\, \mathrm{d}r)$ is $\varepsilon \mathrm{d}m$.
- Consider the luminosity (energy per second) L that carries energy away. In the layer, there should be a balance between energy gains and energy losses (plus and minus denotes outgoing from center or toward center, respectively):

$$\varepsilon \mathrm{d}m + L_r^+ + L_{r+\mathrm{d}r}^- = L_r^- + L_{r+\mathrm{d}r}^+ \tag{1.1}$$

$$\varepsilon \mathrm{d}m = [L_{r+\mathrm{d}r}^+ - L_{r+\mathrm{d}r}^-] - [L_r^+ - L_r^-] = L_{r+\mathrm{d}r} - L_r \tag{1.2}$$

$$\varepsilon \mathrm{d}m/\mathrm{d}r = \mathrm{d}L/\mathrm{d}r, \tag{1.3}$$

  by dividing by $\mathrm{d}r$ and letting it go to zero (derivative).
- This standard relation will appear many times:

$$\rho = \frac{\mathrm{d}m}{\mathrm{d}V}, \tag{1.4}$$

  or,

$$\mathrm{d}m = 4\pi r^2 \rho\, \mathrm{d}r \tag{1.5}$$

- If $\rho\varepsilon$ is the energy produced per second in each cm^3 of material, then by unit analysis the energy generated per second in $(r,\, \mathrm{d}r)$ is also given by

$$\varepsilon \mathrm{d}m = (4\pi r^2 \mathrm{d}r)\rho\varepsilon. \tag{1.6}$$

- Therefore, we can finally state

$$\frac{\mathrm{d}L}{\mathrm{d}r} = 4\pi r^2 \rho\, \varepsilon. \tag{1.7}$$

- Sometimes we will use m instead of r as the independent coordinate, and so

$$\frac{\mathrm{d}L}{\mathrm{d}m} = \varepsilon. \tag{1.8}$$

- This is one of the main equations of stellar structure.
- Note that in regions where $\varepsilon = 0$, the luminosity is **constant**.
- The properties of ε, its temperature dependence and derivation, are discussed in Unit 3.
- See Problem 1.1.

1.2 Local thermodynamic equilibrium

- Collisions between particles in a gas and/or radiation allow equilibrium to occur if the distance particles travel and the time between collisions is small compared to macroscopic length and time scales.
- If this condition is met by radiation, it is known as blackbody radiation and the gas and radiation field are at the same temperature locally.
- This is known as *local thermodynamic equilibrium* (LTE).
- Specifically, the mean free path of photons is given by

$$\ell = \frac{1}{\kappa \rho}, \tag{1.9}$$

  where κ is the opacity and ρ is density.
- We'll see later that for a fully ionized gas that electron scattering gives $\kappa = 0.4\,\mathrm{cm}^2\,\mathrm{g}^{-1}$.
- Even for an average stellar density of $\rho = 1.4\,\mathrm{g\,cm}^{-3}$, the mean free path of photons is about 1 cm. Lots of collisions.
- In stellar interiors this is almost always the case. Above stellar photospheres this assumption breaks down.
- For example, the radiation leaving the Sun's surface is at about 6,000K. However, the gas (electron) temperature of the corona, through which this radiation passes, can be over 1,000,000K. The matter and radiation have not equilibrated. This is actually an outstanding problem.
- On the other end, the 6000K radiation from the Sun passes through Earth's 300K atmosphere also without (thankfully) equilibrating.
- In any case, assuming LTE allows us to calculate the interior structure of a star and all the thermodynamic properties in terms of temperature, density, and composition. This is done at each radial location, and as a function of time.

1.3 Are stars a one-fluid plasma?

- We know stars are mostly ionized. Shouldn't we treat the positive and negative charges as separately?
- If collisions are sufficiently frequent and Maxwellian and have times scales much less than other time scales of interest, then we can use a one-fluid model and generally ignore charge separation.
- The Debye length is the length over which an appreciable electric field can arise. It is roughly a measure of the thermal energy/electric potential energy ratio. If this length is short with respect to the plasma, we can assume charge neutrality.

$$\lambda_D = \sqrt{\frac{\epsilon_0 k_\mathrm{B} T_e}{n_e q_e^2}} \approx 6.9 \left(\frac{T_e}{n_e}\right)^{1/2} \times 10 \text{ m} \tag{1.10}$$

- In the core of the Sun, $n_e = 10^{32}$ m^{-3} and $T_e = 10^7$K, so that $\lambda_D \approx 10^{-11}$ m. In the photosphere, $n_e = 10^{11}$ m^{-3} and $T_e = 5 \times 10^3$ K, so that $\lambda_D \approx 1.5 \times 10^{-3}$ m. For the corona, we may find that $\lambda_D \approx 10$ m.
- So the mean free path of ions (electrons) are much smaller than the scale of the variations of physical quantities. Therefore we can treat most regions of stars as a one-fluid plasma system and use hydrodynamics. In coronae, however, it may be necessary to resort to *plasma physics* where many approximations are no longer valid.

1.4 Simple time scales of stars

While stars are for the most part static or quasistatic, and in equilibrium, there are time scales over which change may occur on a global scale. We can consider some quantity, say ϕ, and its rate of change $\dot{\phi} = \mathrm{d}\phi/\mathrm{d}t$. Any relevant time scale is thus $\phi/\dot{\phi}$.

1.4.1 Dynamical timescale

- Let's first consider the smallest (and most observable) timescale $t_{\rm dyn}$.
- Consider a change in the fundamental dimension of the star, its radius $\phi = R$, may be possible to examine.
- Since gravity is the binding force, the velocity in a gravitational field is the escape velocity $\dot{\phi} = (2GM/R)^{-1/2}$.
- Then (neglecting factors of order unity),

$$t_{\rm dyn} = \left(\frac{R^3}{GM}\right)^{1/2}. \tag{1.11}$$

  Note that in terms of the mean density of a star $t_{\rm dyn} \approx (G\,\overline{\rho})^{-1/2}$.
- In terms of solar values, we find

$$t_{\rm dyn} \approx 30\,\mathrm{min}\left(\frac{R}{R_\odot}\right)^{3/2}\left(\frac{M}{M_\odot}\right)^{-1/2}. \tag{1.12}$$

  So since we don't see large-scale changes on such time scales, we know there must be some balance of forces in the Sun. See Problem 1.2.
- Another way of thinking about this is to start with

$$g = \frac{GM}{R^2}. \tag{1.13}$$

- The time for a particle to fall under gravity is $\ell = 1/2gt^2 \rightarrow t = \sqrt{2\ell/g}$. The time scale therefore for a particle to fall, say, a distance $\ell = R/2$ in a star is

$$t_{\rm dyn} = \left(\frac{R^3}{GM}\right)^{1/2}, \tag{1.14}$$

  which reproduces Eq. (1.11).
- Dynamical processes in stars that help it adjust out of hydrostatic equilibrium typically occur over dynamical timescales, such as oscillations and even supernovae.

1.4.2 Thermal timescale

- There exist thermal processes that affect the internal energy of a star, which we'll denote as $\phi = U$.
- As we'll see from the Virial Theorem, $U \approx GM^2/R$.
- The energy changes due to radiation, whose rate of change is the luminosity $\dot{\phi} = L$.
- If we consider that losing its gravitational potential energy is the only real source of energy, then we can calculate the time a star can radiate at a given luminosity. This is the Kelvin-Helmholtz timescale and can be shown to be:
$$t_{\rm KH} \approx 30\,{\rm Myr} \left(\frac{M}{M_\odot}\right)^2 \left(\frac{R}{R_\odot}\right)^{-1} \left(\frac{L}{L_\odot}\right)^{-1}. \tag{1.15}$$
- See Problem 1.3
- If a star has no internal energy sources it can generate energy and radiate by contracting. You see this in Computer Problem 1.1.
- In Lord Kelvin's time, this was a problem because we assumed the Sun would be older than this value, since we by then knew the Earth to be at least several billion years old. However, we still didn't know about nuclear energy sources.

1.4.3 Nuclear timescale

- Unit 3 discusses the fusion of hydrogen into helium, where the energy released can be estimated as $\Delta E = \Delta mc^2$. We know about 0.7% of the mass is lost.
- If this fusion process only takes place in the inner 10% of the Sun, the energy available is about $7 \times 10^{-4} Mc^2 = \phi$.
- The rate of change of the energy is again the luminosity $\dot{\phi} = L$.
- The timescale is
$$t_{\rm nuc} = 7 \times 10^{-4} \frac{Mc^2}{L} \tag{1.16}$$
$$= 10^{10}\,{\rm yr} \left(\frac{M}{M_\odot}\right) \left(\frac{L}{L_\odot}\right)^{-1}. \tag{1.17}$$

  We know that luminosity is a strong function of mass, so massive stars burn out their energy very quickly.

1.5 Problems

PROBLEM 1.1: [5 pts]: Verify the statement that where $\varepsilon = 0$ the luminosity is constant using `MESA`, by showing an appropriate/convincing plot from a model.

PROBLEM 1.2: [5 pts]: Derive equation (1.12) by plugging in the constants. Qualitatively, how would the dynamical times scale for a white dwarf star compare to the Sun? A supergiant star?

PROBLEM 1.3: [5 pts]: Derive equation (1.15).

COMPUTER PROBLEM 1.1: [25 pts]: Here you will look at the effects of "turning off" nuclear reactions at the main sequence to see how stellar evolution changes. `MESA` allows one full control of nuclear energy generation. We will explore how this changes the star right after the time it formed and is getting to the main sequence.
Submit your answers to the questions below with figures. You may also prepare a document with all answers and figures and upload into Canvas.

What to do

1. Copy the `WORK_DIR` to wherever you will be running MESA, rename it to something sensible.
2. Edit the `inlist_project` file. In `&star_job`, make sure `pgstar_flag=.true.` and we don't need to create a pre-main sequence track yet, so you can add `create_pre_main_sequence_model=.false.` In the `&controls` section, we want the `initial_mass=1.0`. Run the model until about 10 billion years, so set `max_age=1d10`. Most importantly, for this first run, turn off the nuclear reaction rates by setting `eps_nuc_factor=0` and `dxdt_nuc_factor=0`.
3. You shouldn't need to run more than about 1000 models (timesteps) to reach that age based on the default dt. All the data gets saved in `LOGS/`.
4. Now copy a new working directory and maybe copy the inlist you just used into it and change the following: Turn on the reactions by setting that variable to 1. We need a stopping criterion, because 10 billion years would take us to the terminal age main sequence, and so we'll use the onset of hyrdrogen burning and set an abundance criterion. So in `&controls` add `xa_central_lower_limit_species(1)='h1'` and then `xa_central_lower_limit(1)=0.69` (the default initial H abundance is 0.7). So right after a little bit of central hydrogen is burned (depleted), the simulation will stop.

Questions

1. How old was the star with nuclear burning when the hydrogen abundance dropped below 0.7? Is that reasonable?
2. Plot a proper HR diagram with the "tracks" of both stars on it (luminosity vs effective temperature, logarithmically). Try to give some indication of age on the plot. You may have to "zoom" in to the appropriate area with a second plot.
3. Describe the two tracks qualitatively.
4. How long does it roughly take for the model with no nuclear burning to get to its highest $T_{\rm eff}$. How does that number compare with what is predicted from Equation (1.15). Please show your work.
5. Explain why or how the star with no nuclear burning gets so much hotter than the star with nuclear burning. What is physically happening? Then show a plot that should confirm your explanation that uses interior parameters. What happens for the star with no burning at later times, explain its track on the HR diagram? (Look around for the appropriate quantities to plot in the evolution variables, it's up to you).

What You Should Know How To Do From This Chapter

- Know how to put expressions into convenient forms with the proper coefficient as in the timescale equations.