is called the source function, which has the units of specific intensity [erg cm^{-2} rad^{-2} Å^{-1} sec^{-1}]. The source function is a convenient quantity for describing the net specific intensity emitted into the medium at location $\mathbf{x}$ and time t under general thermodynamic conditions.

3.4 The transfer equation

Define $\Delta\epsilon_\lambda$ as the difference between the radiative energy in the wavelength interval $\lambda \to \lambda + d\lambda$ at position $\mathbf{x}$ and time t and the radiative energy in the wavelength interval $\lambda \to \lambda + d\lambda$ propagating in direction $\hat{\mathbf{s}}$ that emerges into solid angle $d\Omega$ across area dA at position $\mathbf{x}+\mathbf{ds}$ and time $t+dt$. This difference is equivalent to the amount of radiative energy created by emission processes (Eq. 3.47) less the amount absorbed (Eq. 3.42) in the volume element $dA\,ds$,

$$\Delta\epsilon_\lambda = [\eta_\lambda(\mathbf{x};t) - \chi_\lambda(\mathbf{x};t) I_\lambda(\mathbf{x};t)]\; dA\, d\Omega\, d\lambda\, dt\, ds. \tag{3.51}$$

Invoking the definition of the specific intensity (Eq. 3.9), the quantity $\Delta\epsilon_\lambda$ can be written

$$\begin{aligned} \Delta\epsilon_\lambda &= d\epsilon_\lambda(\mathbf{x}+\mathbf{ds};t+dt) - d\epsilon_\lambda(\mathbf{x};t) \\ &= [I_\lambda(\mathbf{x}+\mathbf{ds};t+dt) - I_\lambda(\mathbf{x};t)]\; dA\, d\Omega\, d\lambda\, dt \\ &= \left[\frac{1}{c}\frac{\partial I_\lambda(\mathbf{x};t)}{\partial t} + \frac{\partial I_\lambda(\mathbf{x};t)}{\partial s}\right] ds\, dA\, d\Omega\, d\lambda\, dt, \end{aligned} \tag{3.52}$$

where the last step follows from finite differencing,

$$\frac{1}{c}\frac{\partial I_\lambda(\mathbf{x};t)}{\partial t} + \frac{\partial I_\lambda(\mathbf{x};t)}{\partial s} = \frac{I_\lambda(\mathbf{x}+\mathbf{ds};t+dt) - I_\lambda(\mathbf{x};t)}{ds}. \tag{3.53}$$

A schematic of the finite difference expressed in Eq. 3.52 is illustrated in Figure 3.5. Note that for purposes of describing the modification of a beam as it propagates in a fixed direction, the volume element of material, $dA\,ds$, is oriented perpendicular to the propagation direction; that is, $\hat{\mathbf{s}} = \hat{\mathbf{n}}$.

The general expression for the spatial portion of the partial derivative is

$$\frac{\partial I_\lambda(\mathbf{x};t)}{\partial s} = \hat{\mathbf{s}} \cdot \nabla I_\lambda(\mathbf{x};t). \tag{3.54}$$

Equating Eqs. 3.51 and 3.52, and substituting Eq. 3.54, we have the generalized equation of radiative transfer

$$\frac{1}{c}\frac{\partial I_\lambda(\mathbf{x};t)}{\partial t} + \hat{\mathbf{s}} \cdot \nabla I_\lambda(\mathbf{x};t) = \eta_\lambda(\mathbf{x};t) - \chi_\lambda(\mathbf{x};t) I_\lambda(\mathbf{x};t). \tag{3.55}$$

Dividing by the extinction coefficient (which is the equivalent of multiplying by the mean free path), we obtain

$$\frac{1}{\chi_\lambda(\mathbf{x};t)}\left[\frac{1}{c}\frac{\partial I_\lambda(\mathbf{x};t)}{\partial t} + \hat{\mathbf{s}} \cdot \nabla I_\lambda(\mathbf{x};t)\right] = S_\lambda(\mathbf{x};t) - I_\lambda(\mathbf{x};t), \tag{3.56}$$

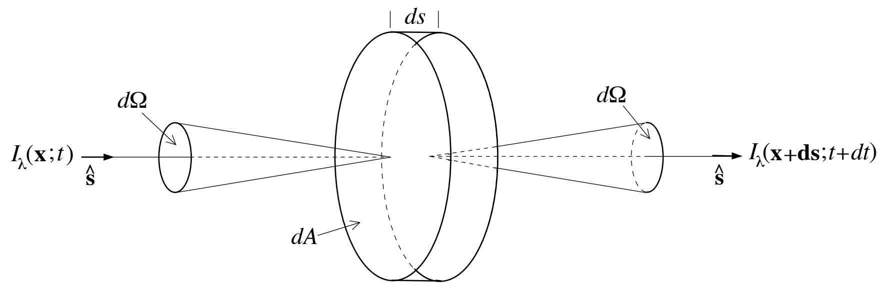

Figure 3.5: A schematic of the absorbing and emitting material and the geometric configuration for the equation of transfer. In a time interval dt, an incident beam of specific intensity $I_\lambda(\mathbf{x};t)$ propagating in the $\hat{\mathbf{s}}$ direction within solid angle $d\Omega$ travels a distance $ds = c \cdot dt$ through a volume element of cross sectional area dA. Following absorption, scattering, and/or emission processes due to interactions with the material, the beam exits at position $\mathbf{x} + \mathbf{ds}$ with modified specific intensity $I_\lambda(\mathbf{x} + \mathbf{ds}; t + dt)$ into the same solid angle $d\Omega$.

where $S_\lambda(\mathbf{x};t)$ is the source function (Eq. 3.50).

In Cartesian coordinates, we have

$$\hat{\mathbf{s}} \cdot \nabla I_\lambda(\mathbf{x};t) = \cos\alpha \frac{\partial I_\lambda(\mathbf{x};t)}{\partial x} + \cos\beta \frac{\partial I_\lambda(\mathbf{x};t)}{\partial y} + \cos\gamma \frac{\partial I_\lambda(\mathbf{x};t)}{\partial z} \tag{3.57}$$

where $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are the direction cosines (Eq. 3.2) of the propagation direction $\hat{\mathbf{s}}$. In spherical coordinates, we have

$$\hat{\mathbf{s}} \cdot \nabla I_\lambda(\mathbf{x};t) = \frac{\partial I_\lambda(\mathbf{x};t)}{\partial r}(\hat{\mathbf{s}} \cdot \hat{\mathbf{r}}) + \frac{\partial I_\lambda(\mathbf{x};t)}{r\,\partial\theta}(\hat{\mathbf{s}} \cdot \hat{\mathbf{\Theta}}) + \frac{\partial I_\lambda(\mathbf{x};t)}{r\sin\theta\,\partial\phi}(\hat{\mathbf{s}} \cdot \hat{\mathbf{\Phi}}). \tag{3.58}$$

3.4.1 Applying plane parallel geometry

In the scenario of quasar absorption line studies of intervening absorbers, the only known geometric information is the line of sight direction to the quasar. There is no *a priori* knowledge of the geometry of an absorber in the case of an intergalactic or individual galactic halo "cloud". It is common practice to assume a plane parallel geometry.

For purposes of illustration, assume a steady state (equilibrium) condition in a plane parallel one dimensional gas "cloud" of thickness L. Let the depth into the cloud be measured by the coordinate z in the Cartesian system, where $z = 0$ is the cloud "face". Further, assume the photon propagation direction (line of sight being measured) is in the $\hat{\mathbf{s}}$ direction, such that $\hat{\mathbf{s}} \cdot \hat{\mathbf{k}} = \cos\theta$, where θ is the angle between $\hat{\mathbf{s}}$ and the coordinate direction $\hat{\mathbf{k}}$. A schematic of the scenario is illustrated in Figure 3.6. A beam incident on the cloud face has specific intensity $I_\lambda(0)$. In order to obtain a general form of the solution to the transfer equation, we will assume that emission, $S_\lambda(z)$, occurs as a general function of z within the cloud.

The equation of transfer for this scenario is written

$$\frac{1}{\chi_\lambda(z)}\left[(\hat{\mathbf{s}}\cdot\hat{\mathbf{k}})\,\frac{dI_\lambda(z)}{dz}\right] = S_\lambda(z) - I_\lambda(z). \tag{3.59}$$

As written, a simple analytical solution for $I_\lambda(z)$ is impossible because the integrand, $\chi_\lambda(z)[S_\lambda(z) - I_\lambda(z)]\,dz/\cos\theta$, is undetermined. Invoking the definition of the absorptivity and accounting for the propagation direction with respect to the geometric coordinate, we have $d\tau_\lambda = \chi_\lambda(z)\,ds = \chi_\lambda(z)\,dz/\cos\theta$. Thus, as illustrated in Figure 3.61, the optical depth along the photon propagation vector can be written in terms of the physical depth[4]. From Eq. 3.46, we have

$$\tau_\lambda(z) = \frac{1}{\cos\theta}\int_0^z \chi_\lambda(z')\,dz', \tag{3.60}$$

where we have explicitly included the z dependence of the optical depth.

We thus can rewrite the transfer equation as a function of the optical depth as the independent variable,

$$\frac{dI_\lambda(\tau_\lambda)}{d\tau_\lambda} = S_\lambda(\tau_\lambda) - I_\lambda(\tau_\lambda). \tag{3.61}$$

3.4.2 Solution in one dimension

The form of the transfer equation, written as Eq. 3.61, is now such that we can apply the standard integrating factor technique and assume a solution of the form,

$$I_\lambda(\tau_\lambda) = Q_\lambda(\tau_\lambda)\cdot\exp\left\{p\tau_\lambda\right\}. \tag{3.62}$$

Differentiating, we have

$$\frac{dI_\lambda(\tau_\lambda)}{d\tau_\lambda} = \frac{dQ_\lambda(\tau_\lambda)}{d\tau_\lambda}\exp\left\{p\tau_\lambda\right\} + p\,Q_\lambda(\tau_\lambda)\cdot\exp\left\{p\tau_\lambda\right\}. \tag{3.63}$$

Equating Eqs. 3.61 and 3.63, we have the relations

$$\begin{aligned} S_\lambda(\tau_\lambda) &= \frac{dQ_\lambda(\tau_\lambda)}{d\tau_\lambda}\exp\left\{p\tau_\lambda\right\} \\ -I_\lambda(\tau_\lambda) &= p\,Q_\lambda(\tau_\lambda)\cdot\exp\left\{p\tau_\lambda\right\}. \end{aligned} \tag{3.64}$$

[4]As mentioned above, the distinction of a photon propagation (line of sight) direction different than the coordinate direction is not commonly invoked for intervening quasar absorption line systems, i.e., $\cos\theta = 1$. We introduce the distinction simply for illustrative purpose. Alternative definitions, such as for stellar atmosphere work, do not incorporate the direction cosine into the definition of the optical depth, but leave it as a geometric factor in the transfer equation, i.e., $d\tau_\lambda = \chi_\lambda(z)\,dz$. This definition provides a "projected" optical depth, which is useful where optical depth changes as a function of viewing angle on the stellar disk for a fixed physical depth into the atmosphere.

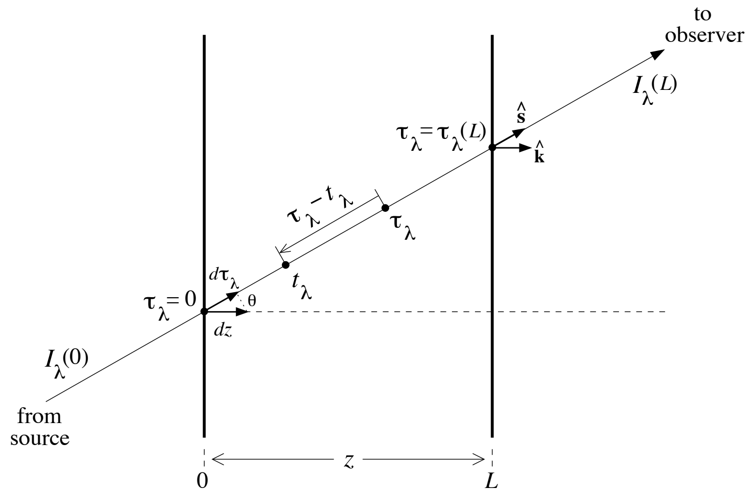

Figure 3.6: Schematic of a one dimensional plane parallel "cloud" of thickness L through which a pencil beam of radiative energy of specific intensity $I_\lambda(z)$ is propagating in the $\hat{\mathbf{s}}$ direction such that $\hat{\mathbf{s}} \cdot \hat{\mathbf{k}} = \cos\theta$, where $\hat{\mathbf{k}}$ is the unit vector along the geometric axis of the cloud. The specific intensity at an optical depth τ_λ measured from the cloud "face" ($\tau_\lambda = 0$) is the sum of the extinction $I_\lambda(0)\exp\{-\tau_\lambda\}$ of the incident specific intensity and the integrated contribution of the extinctions from the emission, $S_\lambda(t_\lambda)$, at each $\tau_\lambda - t_\lambda$, i.e., $\int S_\lambda(t_\lambda)\exp\{-(\tau_\lambda - t_\lambda)\}\, dt_\lambda$.

From Eq. 3.62, the latter relation yields $p = -1$, which after substitution into the former relation for $S_\lambda(\tau_\lambda)$ and solving for $Q_\lambda(\tau_\lambda)$ yields the integral

$$Q(\tau_\lambda) = C + \int_0^{\tau_\lambda} S_\lambda(t_\lambda) \cdot \exp\{t_\lambda\}\, dt_\lambda. \tag{3.65}$$

Substituting Eq. 3.65 into Eq. 3.62 gives

$$I_\lambda(\tau_\lambda) = C\exp\{-\tau_\lambda\} + \exp\{-\tau_\lambda\}\int_0^{\tau_\lambda} S_\lambda(t_\lambda)\exp\{t_\lambda\}\, dt_\lambda. \tag{3.66}$$

Evaluating at the boundary $\tau_\lambda = 0$ where the specific intensity incident upon the cloud face is $I_\lambda(0)$, we have $C = I_\lambda(0)$. Following substitution, and rearranging the second term, the solution to the transfer equation is written

$$I_\lambda(\tau_\lambda) = I_\lambda(0)\exp\{-\tau_\lambda\} + \int_0^{\tau_\lambda} S_\lambda(t_\lambda)\exp\{-(\tau_\lambda - t_\lambda)\}\, dt_\lambda. \tag{3.67}$$

3.4.3 Interpreting the solution: simple cases

Employing the optical depth as the dependent variable in Eq. 3.67 may seem somewhat less intuitive than solving the transfer equation as a function of physical depth. In fact, note that τ_λ is an integral from the cloud face to the corresponding path length (or deprojected physical depth), and requires knowledge of the extinction coefficient as a function of physical depth. However, in practice, the transfer equation is simply a tool applied to observational spectra, which directly provide the optical depth as a function of wavelength, λ. We discuss this further in § 3.7

The λ subscripts denote that Eq. 3.67 is written for radiation in the wavelength range $\lambda \to \lambda + d\lambda$. If the extinction coefficient is wavelength dependent, then the physical depth corresponding to a given optical depth will also be wavelength dependent. This means that the average photon observed at given wavelength will originate from a different depth in the cloud than an average photon at a different wavelength.

Interpreting Eq. 3.67, it is clear that two components govern the behavior of the specific intensity as a function of optical depth (or physical depth) along a line of sight. In the case that there is no emission within the cloud, $S_\lambda(\tau_\lambda) = 0$ (an absorbing cloud), then

$$I_\lambda(\tau_\lambda) = I_\lambda(0) \exp\{-\tau_\lambda\} . \tag{3.68}$$

For heuristic illustration, assume the extinction coefficient is a path length averaged value, which we will denote $\bar{\chi}_\lambda$, and is constant throughout the cloud. Then $\tau_\lambda(s) = \bar{\chi}_\lambda \, s$, were $s = z/\cos\theta$ is the coordinate position along the line of sight that the photons have traveled through the cloud medium. For $\bar{\chi}_\lambda$ constant, Eq. 3.68 can be written as a function of the path length s,

$$I_\lambda(s) = I_\lambda(0) \exp\{-\bar{\chi}_\lambda \, s\} . \tag{3.69}$$

When $s = \bar{\chi}_\lambda^{-1}$, the photons have traveled one complete mean free path and the specific intensity has suffered extinction by the factor $e^{-1} = 0.367$. For every multiple additional mean free path the beam travels, the extinction is an additional factor of 0.367. If a beam travels a distance $s = L$ through an absorbing cloud, then the total optical depth of the cloud is $\tau_\lambda = \bar{\chi}_\lambda \, L$. That is, the total optical depth of a cloud of total path length L is directly proportional to the magnitude of the extinction coefficient.

In Figure 3.7, the behavior of Eq. 3.69 is illustrated as a function of path length for four different extinction coefficients in an absorbing cloud through which the total path length is L. The different extinction coefficients result in different extinction rates and therefore in different total optical depths, τ_λ. Four cases are shown, $\tau_\lambda = 0.5$, 1.0, 2.0, and 4.0. The specific intensity of the incident beam is $I_\lambda(0)$. The ratios of the specific intensity of the emerging beam to the incident beam, $I_\lambda(s \geq L)/I_\lambda(0)$, are given for the four total optical depths. Eq. 3.68 is the workhorse expression for analysis of quasar absorption line data. That is, in most all observational situations, emission into the line of sight to the observer is so negligible that the source function can be omitted from the radiative transfer.

In the case where the source function is not negligible, the second term of Eq. 3.67 accounts for specific intensity added to the beam at different locations along the line of sight, s. If the source function is a constant, S_λ, throughout the cloud, then integration of Eq. 3.67 yields

$$I_\lambda(s) = I_\lambda(0) \exp\{-\bar{\chi}_\lambda \, s\} + S_\lambda \cdot \Big[1 - \exp\{-\bar{\chi}_\lambda \, s\} \Big], \tag{3.70}$$

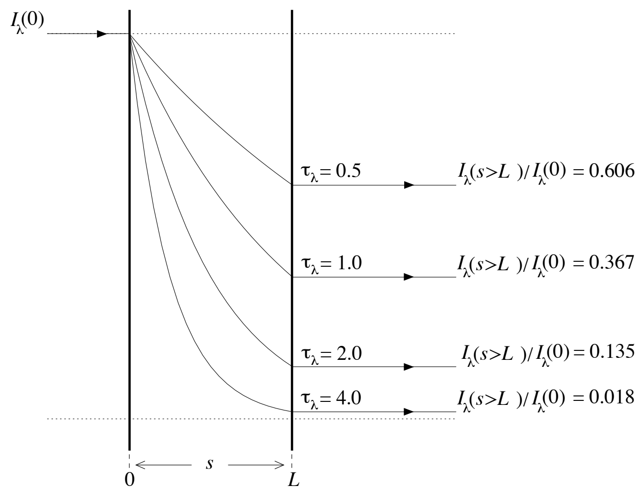

Figure 3.7: A schematic of radiative transfer through an absorbing cloud, i.e., $S_\lambda(\tau_\lambda) = 0$, of total path length L, for which the total optical depth is $\tau_\lambda = 0.5$, 1.0, 2.0, and 4.0. The incoming beam $I_\lambda(0)$ (propagating from left to right) suffers extinction through the cloud according to Eq. 3.69. Since τ_λ is the integral of $\bar{\chi}_\lambda$ over the path length from $0 \leq s \leq L$, the examples are for different opacities (extinction coefficients).

where, following the integration, we again assumed $\tau_\lambda = \bar{\chi}_\lambda\, s$. Eq. 3.70 is illustrated in Figure 3.8*a* with $S_\lambda = 0.25 I_\lambda(0)$ for total optical depths $\tau_\lambda = 1.0$ and 4.0.

The most general case is that the source function is a smooth function of physical depth. In the case of a strong discontinuity in the source function at some location in the cloud, the behavior of $I_\lambda(s)$ can be approximated by treating the cloud as two adjoining media. For example, consider a scenario in which $S_\lambda = 0$ for $0 \leq s < l$ and $S_\lambda =$ constant for $l \leq s \leq L$, where $0 \leq l \leq L$. We will assume $\bar{\chi}_\lambda$ is the same in both "media", but this assumption is not required. The specific intensity will suffer extinction according to Eq. 3.69 until $s = l$, at which point $I_\lambda(l) = I_\lambda(0) \exp\{-\bar{\chi}_\lambda\, l\}$. For the remaining path length $l \leq s \leq L$, the specific intensity will be

$$\begin{aligned} I_\lambda(s) &= I_\lambda(l) \exp\{-\bar{\chi}_\lambda\,(s-l)\} + S_\lambda \cdot \Big[1 - \exp\{-\bar{\chi}_\lambda(s-l)\} \Big] \\ &= I_\lambda(0) \exp\{-\bar{\chi}_\lambda\, s\} + S_\lambda \cdot \Big[1 - \exp\{-\bar{\chi}_\lambda(s-l)\} \Big], \end{aligned} \tag{3.71}$$

where the second form is obtained following substitution of $I_\lambda(l)$. Eq. 3.71 is illustrated in Figure 3.8*b* for $l = L/2$ and $S_\lambda = 0.25 I_\lambda(0)$ for total optical depths $\tau_\lambda = 1.0$ and 4.0. Note that in the case of $\tau_\lambda = 4.0$, the specific intensity quickly converges to the value of the source function.

It is the general behavior that when $S_\lambda \neq 0$, $I_\lambda(s)$ converges toward the value S_λ with increasing path length, s, through the cloud. The convergence

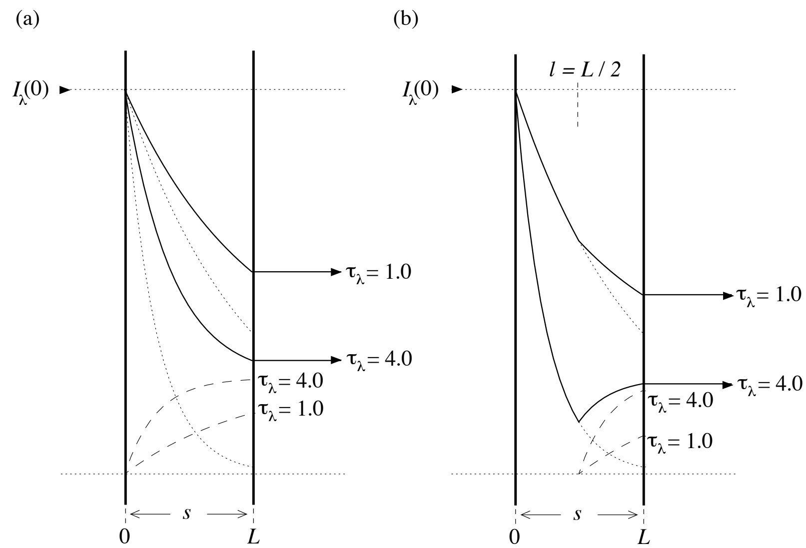

Figure 3.8: Schematics of radiative transfer through a cloud of total path length L with constant source function, $S_\lambda = 0.25 I_\lambda(0)$, where $I_\lambda(0)$ is the specific intensity incident on the cloud face. Examples with total optical depths $\tau_\lambda = 1.0$, and 4.0 are shown. (*a*) The source function is a constant over the full path length $0 \leq s \leq L$. Solid curves are $I_\lambda(s)$ given by Eq. 3.70, dotted curves are the first term, $I_\lambda(0) \exp\{-\bar{\chi}_\lambda\, s\}$, and dashed curves are the second term $S_\lambda \cdot [1 - \exp\{-\bar{\chi}_\lambda\, s\}]$. (*b*) The source function is $S_\lambda = 0$ for $s < l$ and $S_\lambda = 0.25 I_\lambda(0)$ for $l \leq s \leq L$, where $l = L/2$ for this example. $I_\lambda(s)$ obeys Eq. 3.69 for $s < l$ and Eq. 3.71 for $l \leq s \leq L$. The curves represent the same terms as in panel *a*.

(or path length required for convergence) depends upon both the magnitude of the absorption coefficient and the ratio $I_\lambda(s)/S_\lambda$ at the boundary layer where $S_\lambda \neq 0$. For larger $\bar{\chi}_\lambda$, the *rate* of convergence is more rapid. The further the ratio $I_\lambda(s)/S_\lambda$ is from unity at the boundary, the more path length that is required to converge for a given $\bar{\chi}_\lambda$.

The convergence of S_λ to $I_\lambda(s)$ is directly described by Eq. 3.70. The term $I_\lambda(0) \exp\{-\bar{\chi}_\lambda\, s\}$ ranges from $I_\lambda(0) \rightarrow 0$ and the rate at which it vanishes with increasing path length is more rapid for larger $\bar{\chi}_\lambda$. The term $S_\lambda \cdot [1 - \exp\{-\bar{\chi}_\lambda\, s\}]$ ranges from $0 \rightarrow S_\lambda$ at a rate parallel with the vanishing of the first term. As such, for internally emitting optically thick clouds, which is to say for clouds with a mean ratio $\bar{\chi}_\lambda/L \gg 1$, the incident specific intensity is quickly attenuated, the specific intensity quickly equals the source function throughout the cloud, and the emerging specific intensity from the cloud face will be the mean value of the source function.

3.5 What astronomical spectra record

An astronomical spectrum is a recording of the flux of the source. The observed flux from an unocculted, unresolved spherical source of radius R_s at a distance D