

Chapter 12

Atomic cross sections

The probability that an absorber (atom of a given species in a given excitation state and ionization level) will interact with an incident photon of wavelength λ is quantified by the absorption cross section. Similarly, for scattering, the probability of interaction is quantified by the scattering cross section. Generally, cross sections are written σ . For atomic absorption and scattering, cross sections are often written $\alpha(\lambda)$, where the explicit wavelength dependence is included. The cross section is not the probability of interaction; but the probability of interaction is proportional to the cross section. The cross section is the effective “area of interaction” per absorber at wavelength λ .

For a given absorber, the cross section can vary by orders of magnitude as a function of wavelength, or radiative energy. In the case of bound–bound transitions, the cross section has a narrow and large amplitude peak with wavelength. In the case of bound–free transitions, the cross sections exhibit ionization edges, where the absorbers are effectively invisible for photon energies below the ionization potential of the absorber.

Though the cross section is unrelated to physical size of an absorber, given that the Bohr radius of hydrogen is $\simeq 5.3 \times 10^{-9}$ [cm], the physical cross section of hydrogen is $\simeq 3 \times 10^{-17}$ [cm²].

12.1 The classical oscillator

To compute the power removed *from* a beam by a classical oscillator, we first consider power radiated from an accelerating free electron. This power is the radially directed radiative flux generated by the accelerating charge e of mass m_e integrated over all solid angles. We will then equate the power radiated to the power removed from the incident beam.

12.1.1 Accelerating electron

Consider an electron undergoing acceleration at the origin of a spherical coordinate system, i.e., $\mathbf{x} = (r, \phi, \theta)$, and that the acceleration is parallel to the polar axis (the $\hat{\mathbf{k}}$ direction). The instantaneous electric and magnetic field vectors a distance r from the electron are

$$\begin{aligned}\mathbf{E}_\lambda(\mathbf{x}; t) &= \frac{e}{c^2} \frac{\ddot{x}(t)}{r} \sin \theta \hat{\Theta} \\ \mathbf{B}_\lambda(\mathbf{x}; t) &= \frac{e}{c^2} \frac{\ddot{x}(t)}{r} \sin \theta \hat{\Phi},\end{aligned}\tag{12.1}$$

where the polar angle θ is the angle between the radial vector $\hat{\mathbf{r}} = r \hat{\mathbf{r}}$ and the acceleration vector, $\ddot{\mathbf{x}}(t) = \ddot{x}(t) \hat{\mathbf{k}}$. The instantaneous flux at \mathbf{x} from the accelerating charge is given by the Poynting vector,

$$\mathbf{S}_\lambda(\mathbf{x}; t) = \frac{c}{4\pi} [\mathbf{E}_\lambda(\mathbf{x}; t) \times \mathbf{B}_\lambda(\mathbf{x}; t)] = \frac{e^2}{4\pi c^3} \frac{\ddot{x}^2(t)}{r^2} \sin^2 \theta \hat{\mathbf{r}}.\tag{12.2}$$

and the macroscopic radially directed flux at location \mathbf{x} is the time average of the Poynting vector

$$\mathbf{F}_\lambda(\mathbf{x}; t) = \langle \mathbf{S}(\mathbf{x}; t) \rangle = \frac{e^2}{4\pi c^3} \frac{\sin^2 \theta}{r^2} \langle \ddot{x}^2(t) \rangle \hat{\mathbf{r}}.\tag{12.3}$$

If the acceleration is cyclic, then the computation of the time average is straight forward, as will be described below in § 12.1.2 for a harmonic oscillator.

To obtain the cycle averaged instantaneous monochromatic power radiated over all solid angles (the total power), we employ Eq. ??

$$\mathcal{P}_\lambda(t) = \oint_r \mathbf{F}_\lambda(\mathbf{x}; t) \cdot d\mathbf{A} = \oint_r \langle \mathbf{S}_\lambda(\mathbf{x}; t) \rangle \cdot d\mathbf{A}\tag{12.4}$$

where $d\mathbf{A} = r^2 \sin \theta d\theta d\phi \hat{\mathbf{r}}$. We have

$$\mathcal{P}_\lambda(t) = \frac{e^2}{4\pi c^3} \langle \ddot{x}^2(t) \rangle \int_0^{2\pi} \int_0^\pi \sin^3 \theta d\theta d\phi,\tag{12.5}$$

Using the substitution $\mu = \cos \theta$, so that $\sin^2 \theta = 1 - \mu^2$ and $d\mu = -\sin \theta d\theta$, the integral evaluates to $8\pi/3$, yielding

$$\mathcal{P}_\lambda(t) = \frac{2e^2}{3c^3} \langle \ddot{x}^2(t) \rangle.\tag{12.6}$$

Thus, the total radiative power, i.e., the radially directed radiative flux passing through a surface subtending all 4π steradians of solid angle, generated by an accelerating electron is proportional to the time average of the square of the acceleration. What is required to compute the radiative power is a solution to the equation of motion for the electron.

12.1.2 Simple oscillator

Harmonic oscillation of an electron, a natural result of oscillating electric and magnetic fields (at least in the case of forced oscillations), results in acceleration, $\ddot{x}(t)$, proportional and opposite to the instantaneous displacement, i.e., $-kx(t)$, where k is a constant of proportionality which can be interpreted as the force acting on the electron per unit displacement. Consider a harmonically oscillating electron with zero net forces acting upon it, i.e., a simple harmonic oscillator. Balancing forces, the equation of motion for harmonic oscillation of an electron is

$$\ddot{x}(t) + \omega_0^2 x(t) = 0, \quad (12.7)$$

where $\omega_0^2 = k/m_e$ is the eigenfrequency. Assuming $x(t) = \exp\{\lambda t\}$, we obtain the characteristic polynomial $\lambda^2 + \omega_0^2 = 0$, from which $\lambda = \pm i\omega_0$; there are two general solutions and they are complex conjugates of one another. Employing the superposition principle¹,

$$x(t) = \tilde{C} \exp\{i\omega_0 t\} + \tilde{C}^* \exp\{-i\omega_0 t\}, \quad (12.8)$$

where $\tilde{C} = A + iB$ is complex and $\tilde{C}^* = A - iB$ is the complex conjugate of \tilde{C} , where A is the real part of \tilde{C} and B is the imaginary part. Employing the general identities²

$$\exp\{\pm i\omega_0 t\} = \cos \omega_0 t \pm i \sin \omega_0 t \quad (12.9)$$

we rewrite the solution as $x(t) = 2A \cos \omega_0 t$, where $2A$ is the amplitude, which depends upon the initial conditions. If the electron is at maximum displacement, $2A = x_0$, at $t = 0$, the solution is

$$x(t) = x_0 \cos \omega_0 t. \quad (12.10)$$

To obtain the total radiative power of a simple harmonic oscillating electron, we compute the time average of the acceleration for insertion into Eq. 12.6. The time derivatives of $x(t)$ are $\dot{x}(t) = -\omega_0 x_0 \sin \omega_0 t$ and $\ddot{x}(t) = -\omega_0^2 x_0 \cos \omega_0 t$. Over a single cycle period $\mathcal{T} = 2\pi/\omega_0$, from $t_1 = t - \mathcal{T}/2$ to $t_2 = t + \mathcal{T}/2$, centered on arbitrary time t and separated by exactly a single oscillation cycle such they arise at the same phase in adjacent cycles, we have

$$\langle \ddot{x}^2(t) \rangle = \frac{1}{\mathcal{T}} \int_{t_1}^{t_2} \ddot{x}^2(t) dt = \frac{\omega_0^4 x_0^2}{\mathcal{T}} \int_{t_1}^{t_2} \cos^2 \omega_0 t dt = \frac{\omega_0^4 x_0^2}{2}, \quad (12.11)$$

Thus, the total radiative power of a simple harmonic oscillating electron is

$$\mathcal{P}_\lambda = \frac{2e^2}{3c^3} \langle \ddot{x}^2(t) \rangle = \frac{e^2 \omega_0^4 x_0^2}{3c^3}, \quad (12.12)$$

which is independent of time for steady state oscillations.

¹For linear ordinary differential equations, the form of the solution can be written as the sum of individual solutions.

²We remind the reader that each complex number $\tilde{C} = A + iB$ has a complex conjugate $\tilde{C}^* = A - iB$, such that $C^2 = \tilde{C}\tilde{C}^* = (A + iB)(A - iB) = A^2 + B^2$. Also, a complex number can be written $\tilde{C} = C \exp i\theta = C(\cos \theta + i \sin \theta)$, where $A = C \cos \theta$ and $B = C \sin \theta$. The ratio of the imaginary part to the real part is $B/A = \tan \theta$.

12.1.3 Damped oscillator

Since the accelerating electron is radiating electromagnetic energy, the amplitude of the harmonic oscillations will decay over time. Thus, for the classical oscillator scenario, we can view the electric field generated by the accelerating electron as providing a damping force. The damping is interpreted as a radiation reaction force that is proportional to the instantaneous velocity,

$$\mathbf{F}_\gamma(t) = -m_e\gamma\dot{\mathbf{x}}(t), \quad (12.13)$$

where γ is called the damping constant. The rate at which the oscillations decay and the time dependence of the electron acceleration, which governs the time dependence of the total radiative power by the oscillating electron, all depend upon the magnitude of the damping constant.

The equation of motion with damping is

$$\ddot{x}(t) + \gamma\dot{x}(t) + \omega_0^2x(t) = 0. \quad (12.14)$$

As with the simple harmonic oscillator, if we again assume $x(t) = \exp\{\lambda t\}$, we obtain the characteristic polynomial $\lambda^2 + \gamma\lambda + \omega_0^2 = 0$, which has two roots, $\lambda = -\gamma/2 \pm \sqrt{(\gamma/2)^2 - \omega_0^2}$; there are three general solutions depending upon the ratio $(\gamma/2)/\omega_0$. We consider the “underdamped” case in which $\gamma/2 < \omega_0$, which yields $\lambda = -\gamma/2 \pm i\sqrt{\omega_0^2 - (\gamma/2)^2}$. The underdamped condition is the only one of the three in which oscillations are manifest. We again employ the superposition principle to obtain

$$x(t) = \exp\left\{-\frac{\gamma}{2}t\right\} \left[\tilde{C} \exp\{i\omega t\} + \tilde{C}^* \exp\{-i\omega t\} \right], \quad (12.15)$$

where $\omega = \sqrt{\omega_0^2 - (\gamma/2)^2}$. The real part of the eigenvalues govern the rate of decay and the imaginary part is the angular frequency of the damped oscillation. From Eq. 12.9 and applying the boundary condition of maximum displacement x_0 at $t = 0$, we have

$$x(t) = x_0 \exp\left\{-\frac{\gamma}{2}t\right\} \cos \omega t \quad (12.16)$$

Note that the damped oscillation frequency, ω , is shorter than the eigenfrequency, ω_0 . The electron oscillates with a longer cycle time than the simple harmonic oscillator and the amplitude decays with an e-folding time of $\gamma/2$, where γ is the proportionality constant between $F_\gamma(t)$ and $\dot{x}(t)$ in Eq. 12.13,

We obtain the decay rate (value of the damping constant γ), by recognizing the fact that, averaged over a single cycle, the rate of work done on the electron by the radiation reaction force is equal to the negative of the radiative power loss, $\mathcal{P}_\lambda(t)$,

$$\frac{dW_\gamma}{dt} = \langle \mathbf{F}_\gamma(t) \cdot \dot{\mathbf{x}}(t) \rangle = -\mathcal{P}_\lambda(t), \quad (12.17)$$

where $\mathcal{P}_\lambda(t)$ is taken from Eq. 12.6. The time average of $\mathbf{F}_\gamma(t) \cdot \dot{\mathbf{x}}(t)$ over a single oscillation cycle is

$$\langle \mathbf{F}_\gamma \cdot \dot{\mathbf{x}} \rangle = -m_e\gamma \langle \dot{\mathbf{x}}(t) \cdot \dot{\mathbf{x}}(t) \rangle = -m_e\gamma \frac{1}{T} \int_{t_1}^{t_2} \dot{x}^2(t) dt, \quad (12.18)$$

and the radiative power loss is

$$\mathcal{P}_\lambda(t) = \frac{2e^2}{3c^3} \langle \ddot{x}^2(t) \rangle = \frac{2e^2}{3c^3} \frac{1}{\mathcal{T}} \int_{t_1}^{t_2} \ddot{x}^2(t) dt, \quad (12.19)$$

where $\mathcal{T} = 2\pi/\omega$, and where $t_1 = t - \mathcal{T}/2$ and $t_2 = t + \mathcal{T}/2$ are evaluated at identical phases in the oscillation centered on t .

The remaining steps to obtain the damping constant γ are to compute the first and second time derivatives of Eq. 12.16, perform the integrations in Eqs. 12.18 and 12.19, equate the rate of work done and the power loss (via Eq. 12.17) and solve for γ . Because of the decay of the damped oscillator amplitude with time, the symmetry of $x(t)$, $\dot{x}(t)$, and $\ddot{x}(t)$ for a simple oscillator is broken at times t_1 and $t_2 = t_1 + \mathcal{T}$. Thus, the integrals are non trivial. The mathematics are simplified with little loss of generality by assuming $\gamma \ll \omega_0$. Under this condition the damped oscillation frequency, ω , can be approximated by the eigenfrequency, ω_0 . Moreover, the damping rate of the amplitude is negligible over a single cycle period $\mathcal{T} \simeq 2\pi/\omega_0$. Thus, the solution for the damped oscillator (Eq. 12.16) used to perform the integrations can be approximated using the simple oscillator³ (Eq. 12.10). Applying this assumption, the integral in Eq. 12.18 evaluates to $\omega_0^2 x_0^2/2$ and the integral in Eq. 12.19 evaluates to $\omega_0^4 x_0^2/2$. Equating Eqs. 12.18 and 12.19,

$$-m_e \gamma \frac{\omega_0^2 x_0^2}{2} = -\frac{2e^2}{3c^3} \frac{\omega_0^4 x_0^2}{2}. \quad (12.20)$$

we find the damping constant,

$$\gamma = \frac{2e^2 \omega_0^2}{3m_e c^3} = 6.2664 \times 10^{-24} \omega_0^2 \text{ sec}^{-1}, \quad (12.21)$$

for an underdamped oscillating electron in which its emitted radiation provides a radiative reaction damping force. It is derived assuming the work done by the force is equal to the electrons radiative power loss. Clearly, the assumption that $\gamma \ll \omega_0$ is well founded.

12.1.4 Forced damped oscillator

Consider an electron embedded in a monochromatic plane wave beam in which the oscillating electric field is given by Eq. ???. Placing the electron at $r = 0$, the force acting on the electron is

$$\mathbf{F}_\lambda(t) = e\mathbf{E}_0 \cos \omega t, \quad (12.22)$$

³Most treatments (e.g., Mihalas, 1978; Shu, 1991; Rybicki & Lightman, 2004) carry out the mathematical formalism to show that the radiative reaction force is proportional to $\ddot{x}(t)$, which is difficult to interpret. Following this result, the assumption of a simple harmonic oscillator is *then* applied in order to obtain $\ddot{x}(t) = -\omega^2 \dot{x}(t)$. The reader is referred to other works if such details are sought. Here, to simplify obtaining the result, we begin with the assumption $\gamma \ll \omega_0$ and obtain the identical expression for γ .

where $\omega = 2\pi c/\lambda$ is the angular frequency for photons with wavelength λ . The equation of motion governing the electron is that of a damped harmonic oscillator, but with a driving force

$$\ddot{x}(t) + \gamma\dot{x}(t) + \omega_0^2 x(t) = F(t) = \frac{e}{m_e} E_0 \cos \omega t, \quad (12.23)$$

where the damping force arises from the radiative power loss of the electron (the radiative reaction force discussed in § 12.1.3), with γ given by Eq 12.21.

The solution to Eq. 12.23 has a transitory solution and a steady state solution. The steady state solution is obtained most readily in the complex plane. Since any real quantity can be expressed as a component of a complex quantity (following Eq. 12.9), we can rewrite the driving force, Eq. 12.22, as a real plus imaginary part⁴,

$$F(t) = \tilde{F} \exp\{i\omega t\} = \tilde{F} (\cos \omega t + i \sin \omega t) \quad \tilde{F} = F_0 \exp\{i\phi\}, \quad (12.24)$$

where \tilde{F} is a complex coefficient with amplitude $F_0 = eE_0/m_e$ and phase modulation $\exp\{i\phi\} = \cos \phi + i \sin \phi$. We also write the (yet to be determined) position as real and imaginary parts

$$x(t) = \tilde{x} \exp\{i\omega t\} = \tilde{x} (\cos \omega t + i \sin \omega t) \quad \tilde{x} = x_0 \exp\{i\phi\}, \quad (12.25)$$

where \tilde{x} is complex with amplitude x_0 and phase $\exp\{i\phi\}$. The full expression for the complex solution for the electron motion is

$$x(t) = \tilde{x} \exp\{i\omega t\} = x_0 \exp\{i\phi\} \exp\{i\omega t\} = x_0 \exp\{i(\omega t + \phi)\}. \quad (12.26)$$

The real part of Eq. 12.26 is taken as the solution to Eq. 12.23. Thus,

$$x(t) = x_0 \cos(\omega t + \phi), \quad (12.27)$$

for which we need to determine the amplitude x_0 and phase angle ϕ . From Eq. 12.25, the derivatives of $x(t)$ are

$$\dot{x}(t) = i\omega\tilde{x} \exp\{i\omega t\} \quad \ddot{x}(t) = (i\omega)^2\tilde{x} \exp\{i\omega t\}. \quad (12.28)$$

Substitution into Eq. 12.23 and subsequent cancellation of the $\exp\{i\omega t\}$ terms yields

$$(i\omega)^2\tilde{x} + (i\omega)\gamma\tilde{x} + \omega_0^2\tilde{x} = \tilde{F}. \quad (12.29)$$

Solving for \tilde{x} provides the complex coefficient of $x(t)$,

$$\tilde{x} = \frac{\tilde{F}}{\omega_0^2 - \omega^2 + i\gamma\omega}, \quad (12.30)$$

⁴See footnote 2 of this chapter. A slight difference here is that for ease of solution, we write Eqs. 12.24 and 12.25 as the product two complex numbers (i.e., we multiply by a complex coefficient). This accounts for the fact that the electron motion will not be in phase with the driving force (the complex coefficient is the phase lag term).

which is interpreted as the amplitude of the phase modulation. The electron oscillation amplitude is obtained from $x_0^2 = \tilde{x}\tilde{x}^*$, yielding

$$\begin{aligned}
 x_0 &= \left[\frac{\tilde{F}}{(\omega_0^2 - \omega^2 + i\gamma\omega)} \cdot \frac{\tilde{F}^*}{(\omega_0^2 - \omega^2 - i\gamma\omega)} \right]^{1/2} \\
 &= \left[\frac{F_0 \exp\{i\phi\}}{(\omega_0^2 - \omega^2 + i\gamma\omega)} \cdot \frac{F_0 \exp\{-i\phi\}}{(\omega_0^2 - \omega^2 - i\gamma\omega)} \right]^{1/2} \\
 &= \frac{F_0}{[(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2]^{1/2}} \\
 &= \frac{(e/m_e)E_0}{[(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2]^{1/2}}.
 \end{aligned} \tag{12.31}$$

where \tilde{x}^* and \tilde{F}^* are the complex conjugates of \tilde{x} and \tilde{F} . Note that the amplitude of the electron oscillation depends upon the driving force frequency, ω , and has maximum $eE_0/m_e\gamma\omega_0$ when $\omega = \omega_0$ (a phenomenon called resonance), where E_0 is the amplitude of the oscillating electric field of the incident beam.

Substituting for x_0 in Eq. 12.27, we have

$$x(t) = \frac{(e/m_e)E_0 \cos(\omega t + \phi)}{[(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2]^{1/2}}. \tag{12.32}$$

Thus, we see that, once steady state is achieved, the electron oscillates at the frequency of the electric field of the incident beam. Furthermore, the oscillation is out of phase with the electric field by phase angle ϕ , which can easily be shown to be $\phi = -\tan^{-1}[\gamma\omega/(\omega_0^2 - \omega^2)]$. For underdamped conditions near resonance, the electron lags the electric field by $\phi \simeq \pi/2$.

Following the steps taken in § 12.1.2, we employ Eq. 12.6 to obtain the total power radiated over all solid angles from the time average of the real part of the electron acceleration,

$$\ddot{x}(t) = -\omega^2 x(t). \tag{12.33}$$

Carrying out the time average (see Eq. 12.11), we obtain $\langle \ddot{x}(t) \rangle = \omega^2 x_0^2/2$, from which we find the steady state power of electromagnetic energy emitted by the electron

$$\mathcal{P}_\lambda(\omega) = \frac{e^4\omega^4}{3m_e^2c^3} \frac{E_0^2}{(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2}, \tag{12.34}$$

subject to a monochromatic beam with electric field amplitude E_0 and wavelength $\lambda = 2\pi c/\omega$ (we prefer λ to denote the monochromatic nature of radiation, however, it is presently convenient to write Eq. 12.34 as a function of ω). For $\omega = \omega_0$, the eigenfrequency of the system, the power is a maximum with $\mathcal{P}_{max} = (e^4\omega_0^2/3m_e^2c^3\gamma^2)E_0^2$.

12.1.5 Cross section of the classical oscillator

From the principles of absorption and scattering cross sections discussed in § ??, the total absorption (scattering) cross section per interacting particle at frequency ω is defined as the power absorbed (scattered over all solid angles) from an incident beam normalized to the beam flux,

$$N_T \sigma(\omega) = \frac{\mathcal{P}_\lambda(\omega)}{F_b(\omega)}, \quad (12.35)$$

where we carry over the angular frequency notation for the classical oscillator and explicitly include the monochromatic frequency of the beam flux.

For the following, allow the beam to be polychromatic with the stipulation that the flux is constant at each frequency, i.e., that the specific intensity distribution is constant or the electric field amplitude is constant as a function of ω . As derived in § ??, the flux in the incident beam is then $F_b(\omega) = (c/8\pi)E_0^2$ at all frequencies. As will be demonstrated below, this assumption is not required over a large range of frequencies, but only over an *extremely* narrow range centered on the eigenfrequency ω_0 of the classical oscillator.

We equate the power radiated into all solid angles by the forced accelerating electron (given by Eq. 12.34) as the power scattered out of the incident beam. Barring the radiation scattered back into the direction of the post interaction beam (which is effectively zero since the angular distribution of the scattered power is $d\mathcal{P}(\phi, \theta)/d\Omega \propto \sin^2 \theta$, where $\theta = 0$ in the beam direction), the total power scattered is also equatable to the power absorbed from the beam in a closed system where energy is conserved.

The classical oscillator is a single electron, so that $N_T = 1$. Since the flux in the incident beam is $F_b(\omega) = (c/8\pi)E_0^2$, the frequency dependent total cross section for the classical oscillator is

$$\sigma(\omega) = \frac{\mathcal{P}_\lambda(\omega)}{F_b(\omega)} = \frac{e^4 \omega^4}{3m_e^2 c^3} \frac{E_0^2}{(\omega^2 - \omega_0^2)^2 + \gamma^2 \omega^2} \left(\frac{cE_0^2}{8\pi} \right)^{-1} \quad (12.36)$$

which simplifies to,

$$\sigma(\omega) = \frac{8\pi e^4 \omega^4}{3m_e^2 c^4} \frac{1}{(\omega^2 - \omega_0^2)^2 + \gamma^2 \omega^2}. \quad (12.37)$$

The behavior of Eq. 12.37 is such that the half-height full-width of $\sigma(\omega)$ is $\Delta\omega \simeq \gamma/2$. From Eq. 12.21, $\gamma/\omega_0^2 \simeq 6.3 \times 10^{-24}$, so that the fractional full width is $\Delta\omega/\omega_0 \simeq (\omega_0/2)(\gamma/\omega_0^2) \simeq 3.15 \times 10^{-24}\omega_0$. For ultraviolet and optical transitions, $\omega_0 \simeq 10^{16} \text{ sec}^{-1}$ (for example, neutral hydrogen Ly α has $\omega_0 = 1.549 \times 10^{16} \text{ sec}^{-1}$), yielding $\Delta\omega/\omega_0 \simeq 10^{-8}$. Thus, Eq. 12.37 is highly peaked for the eigenfrequency of atomic transitions (our assumption of a constant flux over this range is thus quite sound). We can thus apply the approximation

$$(\omega^2 - \omega_0^2) = (\omega + \omega_0)(\omega - \omega_0) \simeq 2\omega_0(\omega^2 - \omega_0^2). \quad (12.38)$$

Invoking Eq. 12.38, and replacing all other appearances of ω in Eq. 12.37 with ω_0 , we obtain

$$\sigma(\omega) = \frac{2\pi e^4 \omega_0^2}{3m_e^2 c^4} \frac{1}{(\omega - \omega_0)^2 + (\gamma/2)^2}. \quad (12.39)$$

Since the relationship between the damping constant and the eigenfrequency is $\gamma = 2e^2 \omega_0^2 / 3m_e c^3$, further manipulation yields

$$\sigma(\omega) = \frac{2\pi e^2}{m_e c} \frac{\gamma/2}{(\omega - \omega_0)^2 + (\gamma/2)^2}, \quad (12.40)$$

Within a factor $1/\pi$, the second factor of Eq. 16.3 is known as the Cauchy probability distribution, Breit–Wigner distribution, or more commonly by physicists and astronomers as the Lorentz distribution (or Lorentzian function). Substituting $x = \omega$, and writing $\Delta x = x - x_0$ and also writing $y = \gamma/2$, the Lorentzian is

$$\mathcal{L}(x) = \frac{1}{\pi} \frac{y}{(x - x_0)^2 + y^2} = \frac{1}{\pi} \frac{y}{(\Delta x)^2 + y^2}, \quad (12.41)$$

for which the peak amplitude is $1/\pi y$. Thus, y is known as the scale parameter. The half-width at half-maximum, FWHM/2, is also given by y . The peak is at x_0 , which is known as the location parameter. The area is unity

$$\frac{1}{\pi} \int_0^\infty \frac{y}{(x - x_0)^2 + y^2} dx = \frac{1}{\pi} \int_{-\infty}^\infty \frac{y}{(\Delta x)^2 + y^2} d\Delta x = 1. \quad (12.42)$$

In Figure 12.1, Eq. 12.37 is plotted for various values of γ/ω_0^2 . As the ratio decreases, the function approaches the form of a Lorentzian, such that by $\gamma/\omega_0^2 = 10^{-5}$, the fractional difference between the two functions never exceeds 0.3% within ten half-width at half maximum. For the classical oscillator, $\gamma/\omega_0^2 \simeq 10^{-24}$; the Lorentzian is an excellent approximation for the cross section.

In terms of frequency, we have $\Delta\omega = 2\pi\Delta\nu$, where $\Delta\omega = \omega - \omega_0$ and $\Delta\nu = \nu - \nu_0$; we can thus write Eq. 16.3 as

$$\sigma(\nu) = \frac{e^2}{m_e c} \frac{\gamma/4\pi}{(\nu - \nu_0)^2 + (\gamma/4\pi)^2}, \quad (12.43)$$

for which $y = \gamma/4\pi$. Applying Eq. 16.4 (accounting for the cancelled factor of π), the integrated absorption coefficient over all frequencies is

$$\int_0^\infty \sigma(\nu) d\nu = \int_{-\infty}^\infty \sigma(\Delta\nu) d\Delta\nu = \frac{\pi e^2}{m_e c}. \quad (12.44)$$

In terms of wavelength, we have $\Delta\omega = (2\pi c/\lambda_0^2)\Delta\lambda$, where $\Delta\lambda = \lambda - \lambda_0$; Eq. 16.3 can then be written

$$\sigma(\lambda) = \frac{e^2}{m_e c} \frac{\lambda_0^2}{c} \frac{\gamma\lambda_0^2/4\pi c}{(\lambda - \lambda_0)^2 + (\gamma\lambda_0^2/4\pi c)^2}, \quad (12.45)$$

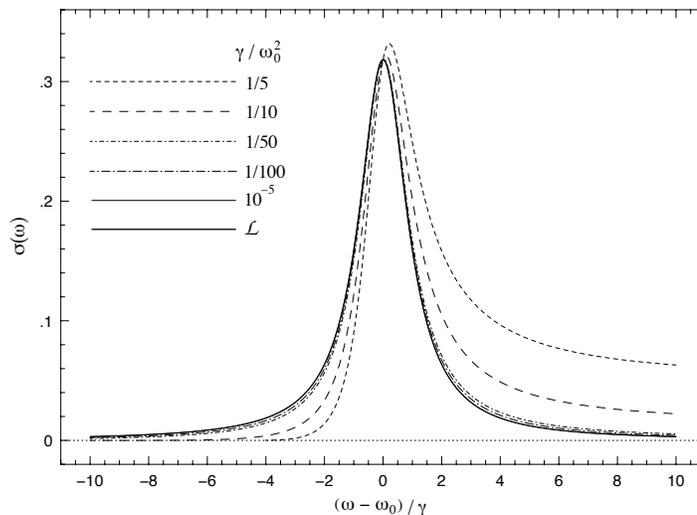


Figure 12.1: A comparison between the full expression of the cross section for the classical oscillator, $\sigma(\omega)$, as given by Eq. 12.37, and the Lorentzian version (designated \mathcal{L} in the figure legend), given by Eq. 12.39 derived from the fact that $\gamma \ll \omega_0^2$. The amplitude of the curves have been normalized for ease of comparison. The shapes are independent of the value of ω_0 ; they depend only upon the ratio γ/ω_0^2 . For $\gamma/\omega_0^2 \leq 10^{-5}$, the two functions differ by no more than 0.0002% at the peak and by no more than 0.3% well into the wings. Since $\gamma/\omega_0^2 \simeq 10^{-24}$ for the classical oscillator, the application of a Lorentzian is well founded.

for which $y = \gamma\lambda_0^2/4\pi c$. The integrated absorption coefficient over all wavelengths is

$$\int_0^\infty \sigma(\lambda) d\lambda = \int_{-\infty}^\infty \sigma(\Delta\lambda) d\Delta\lambda = \frac{\pi e^2}{m_e c} \frac{\lambda_0^2}{c}. \quad (12.46)$$

The integration of $\sigma(\nu)$ over all frequencies and of $\sigma(\lambda)$ over all wavelengths provides the total power removed (or scattered) by the classical oscillator (a single electron subjected to a beam).

12.1.6 Limitations of the classical oscillator

The classical oscillator is a heuristic tool, a model of a free electron subjected to a classical monoenergetic plane-wave electromagnetic beam. The usefulness of the model is that the functional form of the derived cross section, a Lorentzian, is identical to the functional form derived from the full quantum mechanical treatment of atomic bound-bound transitions. However, direct application of the classical oscillator cross section to atomic transitions is not entirely possible due to three limitations. These limitations are due to the quantum mechanical physics of bound-bound transitions that simply cannot be addressed via the classical oscillator scenario.