ASTR 605
Homework 5

1. Bohr postulated a solar system like atom with a constant of motion given by the azimuthal angular momentum of the system, or

$$\int_0^{2\pi} p_\phi \, d\phi = nh, \tag{1}$$

where n is an integer.

(a) Show that

$$p_\phi = m_{\rm N}\omega r_{\rm N}^2 + m_e \omega r_e^2$$

where $\omega = d\phi/dt$, $m_{\rm N}$ is the mass of the nucleus, m_e is the mass of the electron, and where $r_{\rm N}$ and r_e are the radial distances of the nucleus and electron from the center of mass of the system, with the total radial distance between nucleus and electron being $r_n = r_{\rm N} + r_e$.

(b) Given the center of mass relation $m_{\rm N} r_{\rm N} = m_e r_e$, derive

$$\mu\omega r_n^2 = n\hbar \tag{2}$$

from Eq. 1 where $\mu = m_e/(1 + m_e/m_{\rm N})$.

(c) Using the Bohr model energies, $E_n = -R_{\rm Z}/n^2$, calculate the wavelength (in angströms) of the Hβ transition for the Bohr model. Show your calculation.

(d) Consider the Bohr–Sommerfeld model of the atom and the energy levels E_{nk}. (i) How many spectral multiplets are predicted for Hβ transition? (ii) What are the nk combinations for these Hβ transitions, i.e., $nk \leftrightarrow n'k'$? (iii) Are any of the orbits with quantum numbers nk involved in the Hβ multiplet circular? If so, which one(s)? (iii) Compute the wavelengths of the Hβ multiplets for the Bohr–Sommerfeld model.

(e) For which Bohr–Sommerfeld Hβ multiplet is the wavelength closest (least discrepant) to the Bohr wavelength? For which is it furthest (most discrepant)? Comment on the physical reason for these two extreme cases in terms of the physics of Bohr–Sommerfeld orbits?

2. In the first–order relativistic Dirac theory, there are three perturbation correction terms to the Shrödinger Hamiltonian, H (see Eq.10.7 of the class notes).

(a) What are the constants of motion in the Shrödinger model? By what quantum number is each constant of motion represented by? In terms of the wave functions, what does each quantum numbers represent with regard to the stationary states and the wave nature of matter in a bound spherical system.

(b) What are the constants of motion in the approximate Dirac model? What quantum numbers are the represented by?

(c) Equating these perturbation correction terms to the Hamiltonian as H_1, H_2, and H_3, write down expression for each of these three terms.

(d) State what is the *physical* origin of each of these terms. Be as specific as possible (describe as much of the physics as you can– remember quantum numbers only represent physical states).

3. Fine structure levels at $n = 4$.

   (a) Consider Figure 10 of the notes (page 209). Using Equation 10.11 (page 208) extend this diagram to $n = 4$ for all l and j states. [Note the typo on the $2\mathrm{p}_{1/2}$ state, which should read $2\mathrm{p}_{3/2}$]. You do not need to duplicate the $n = 1$, 2, and 3 states, but I do want you to illustrate the $n = 4$ state as precisely as they are presented for the lower states, including the numeric energy differences, $\alpha^{-2}\Delta E_{nj}/E_n$, and quantum states. I suggest either using graph paper or plot this with XFIG to make the energy shifts and splitting in proportion on the paper. How many fine structure multiplets are there for $n = 4$?

   (b) Now duplicate Figure 10.5 *and* add fine structure states for $n = 4$. Be sure to get the energy degeneracies correct and the relative shifts (but do not worry about accuracy of the energy shifts on your diagram, just make them schematically correct).

   (c) For this model: (i) How many allowed dipole fine structure transitions contribute to the Paschen fundamental Pα ($n' = 3, n = 4$). (ii) Draw the allowed dipole transitions on your diagram,and label them A, B, etc). (iii) For each A, B, etc., write the corresponding notation nl_j–$n'l'_{j'}$ (use tabular form). (iv) Which allowed Pα transitions are energy degenerate (using your labels A, B, etc)? (v) How many *spectral* multiplets are there for Pα? (vi) Which potential Pα transitions are forbidden (using nl_j notation) and state which selection rule(s) each would violate?

4. Consider Table 11.2 on page 243 of the notes. Extend this table for all allowed Pα transitions. I will provide the oscillator strength $f_{nl}^{n'l'}$ on a separate page.

5. Higher order corrections?

   (a) What is physical origin of the Lamb shift? What degeneracies are broken by the Lamb shift physics.

   (b) What is the physical origin of hyperfine splitting? Why do the energy splitting depend on the ratio of the electron mass to the nucleus mass.

   (c) Which transition is the famous 21–cm emission line? Is this a forbidden or allowed transition? Explain your answer.