ASTR 545
Homework 4

1. In Homework 3, you calculated P_e/P_g as a function of T for a gas with a single metal (calcium), which has an ionization potential well below that of hydrogen (i.e., $\chi_{\rm M} = 6.1 < \chi_{\rm H} = 13.6$ [eV]). Your calculation showed that $P_e/P_g \ll 1$ for $kT \ll \chi_{\rm H}$ and approached $P_e/P_g \simeq 0.5$ for $kT \gg \chi_{\rm H}$.

   (a) Assuming a gas composed primarily of hydrogen, $\alpha_{\rm H} \simeq 1$, and a metal with $\alpha_{\rm M} \ll \alpha_{\rm H}$, employ the equations of particle and charge conservation under the assumption of single ionizations to show that n_e can be expressed

   $$n_e = n_{\rm H}(f_{\rm H} + \alpha_{\rm M} f_{\rm M}),$$

   and that the ratio of pressures can be expressed

   $$\frac{P_e}{P_g} = \frac{f_{\rm H} + \alpha_{\rm M} f_{\rm M}}{(1 + f_{\rm H}) + \alpha_{\rm M}(1 + f_{\rm M})},$$

   where $f_{\rm H} = N_{1,{\rm H}}/N_{\rm H}$ and $f_{\rm M} = N_{1,{\rm M}}/N_{\rm M}$, respectively.

   (b) For $kT \gg \chi_{\rm H}$, draw upon the temperature dependence (qualitative behavior) of $f_{\rm H}$ and $f_{\rm M}$ to show that

   $$\frac{P_e}{P_g} \simeq \frac{f_{\rm H}}{1 + f_{\rm H}} \simeq 0.5,$$

   and explain your assumptions.

   (c) For $kT \simeq \chi_{\rm H}$, show that

   $$\frac{P_e}{P_g} \simeq f_{\rm H} \ll 1,$$

   and explain your assumptions.

   (d) For $kT \ll \chi_{\rm H}$, show that

   $$\frac{P_e}{P_g} \simeq \alpha_{\rm M} f_{\rm M} \ll 1,$$

   and explain your assumptions.

   (e) Discuss the relevance of metals to the contribution of n_e as a function of temperature (i.e., over what regime do low $\chi_{\rm M}$ metals dominate and over which are they unimportant?). Electron scattering is one source of opacity in stellar atmospheres. In what temperature regime would you expect electron scattering to be important and why?

2. In class, we derived the flux integral

$$F_\nu(\tau_\nu) = 2\pi \left[\int_{\tau_\nu}^{\infty} S_\nu(t_\nu) E_2(t_\nu - \tau_\nu) dt_\nu - \int_0^{\tau_\nu} S_\nu(t_\nu) E_2(\tau_\nu - t_\nu) dt_\nu \right],$$

where

$$E_n(x) = \int_1^{\infty} \frac{e^{-xw}}{w^n} dw$$

is the so–called exponential integral of the nth order. See pages 132–135 in Gray for further information.

(a) Starting with the definition of $J_\nu(\tau_\nu)$, derive the mean intensity integral

$$J_\nu(\tau_\nu) = \frac{1}{2} \left[\int_{\tau_\nu}^{\infty} S_\nu(t_\nu) E_1(t_\nu - \tau_\nu) dt_\nu + \int_0^{\tau_\nu} S_\nu(t_\nu) E_1(\tau_\nu - t_\nu) dt_\nu \right],$$

showing all steps and substitutions. Provide the expression for $J_\nu(0)$ at the stellar surface. Comment on the factor $1/2$ in front of the integral.

(b) Starting with definition of $K_\nu(\tau_\nu)$, derive the "momentum" intensity integral

$$K_\nu(\tau_\nu) = \frac{1}{2} \left[\int_{\tau_\nu}^{\infty} S_\nu(t_\nu) E_3(t_\nu - \tau_\nu) dt_\nu + \int_0^{\tau_\nu} S_\nu(t_\nu) E_3(\tau_\nu - t_\nu) dt_\nu \right],$$

showing all steps and substitutions.

3. Consider a grey stellar atmopshere in which Eddington's first approximation has been assumed

$$\frac{T(\tau)}{T_{\text{eff}}} = \left[\frac{3}{4}\{\tau + q(\tau)\}\right]^{1/4} \equiv \frac{1}{p(\tau)},$$

where $q(\tau)$ is the Hopf function. Assuming radiative equilibrium,

$$S_\nu(\tau) = B_\nu[T(\tau)] = \frac{2h\nu^3}{c^2}\frac{1}{\exp\{h\nu/kT(\tau)\} - 1}$$

(a) Starting with the expression for $F_\nu(\tau)$ from Problem 2, and invoking

$$F = \int_0^\infty F(\tau)d\tau = \sigma T_{\text{eff}}^4,$$

where

$$F(\tau) = \int_0^\infty F_\nu(\tau)d\nu,$$

show that the flux per unit frequency can be written

$$\frac{F_\alpha(\tau)}{F} = \frac{4\pi k^4}{h^3c^2\sigma}\alpha^3\beta(\tau)$$

where

$$\beta(\tau) = \int_\tau^\infty \frac{E_2(t-\tau)dt}{\exp\{\alpha p(t)\} - 1} - \int_0^\tau \frac{E_2(\tau - t)dt}{\exp\{\alpha p(t)\} - 1}$$

(b) Write the expression for the emergent flux at $\tau = 0$, i.e., $F_\alpha(0)/F$.

HINT: Invoke the the defninition

$$\alpha = \frac{h\nu}{kT_{\text{eff}}},$$

which (in order to preserve units) gives

$$F_\alpha(\tau)d\alpha = F_\nu(\tau)d\nu.$$