Components to an Atmosphere’s Gas

1. Radiation Field - distribution of numbers of photon energies of photons

- Thermal Equilibrium - must be locally coupled to thermal conditions of particles (gas)

- Non-Equilibrium - can arise in "distant" region (where it is coupled to thermal properties) and thus influence particle energies and states in a non local way

- Local Thermal Equilibrium (LTE) - thermal conditions of gas and radiation field change on small scales relative to the overall structure; though radiation field not in strict TE with particles it is approximated as being so...

2. Particles - comprising neutral atoms, ionized atoms, free electrons, neutral molecules, and ionized molecules

(stored energy) ⇒ atoms/ions and molecules can have various levels and modes of excitation (stored internal energy)

- Thermal Equilibrium - distribution of electron energies (velocities) and distribution of atomic/molecular excitations, ionizations, and kinetic energies coupled to radiation field.

- Non-LTE and LTE (departure coefficients) - atmosphere not in strict TE or LTE, so theory developed called Non-LTE which starts w/LTE and adds correction factors
Temperature Effects:

1. Distribution of particle speeds
   - Many interactions are velocity dependent.

2. Distribution of photons (numbers and energies)
   - Governs flux off star, continuum radiation field.

3. Distribution of excited states and ionization stages of elements.
   - Thus, governs # of free electrons in gas.
   - Governs strengths of absorption lines.

\[ f(v_x)dv_x = \left\{ \frac{m}{2\pi kT} \right\}^{1/2} \exp \left[ -\frac{mv_x^2}{2kT} \right] dv_x \]

\[ f(v_x) = \frac{n(v_x)}{n} \]

\[ f(v_x)dv_x = \text{number of particles having velocity component in } "x" \text{-direction} \]

\[ m = \text{total number of particles, } n(v_x) = \text{number in range } v \to v+dv \]

\[ m = \text{mass of particle} \]

\[ 3D \]

\[ f(v) = f(v_x)f(v_y)f(v_z) = \left\{ \frac{m^3}{2\pi kT} \right\}^{3/2} \exp \left[ -\frac{m(v_x^2+v_y^2+v_z^2)}{2kT} \right] \]

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(v)dv_xdv_ydv_z = 1 \]

* Thermal broadening of absorption lines with characteristic with

\[ \Delta \lambda_B = \frac{v_0}{c} \lambda = \frac{\lambda}{c} \left\{ \frac{2kT}{m} \right\}^{1/2} \]

\[ V_0 = \text{most probable speed} = \left\{ \frac{2kT}{m} \right\}^{1/2} \]

\[ \text{Average speed} = \left\{ \frac{8kT}{\pi m} \right\}^{1/2} \]

\[ \text{RMS speed} = \left\{ \frac{3kT}{m} \right\}^{1/2} \]

\[ v = \text{velocity dist. in single direction is Gaussian!} \]
Figure 5.5: The Maxwellian distribution of particle speeds and one-dimensional velocity components. (a) The distribution of the observed line of sight component of velocity, $v_\parallel$, for hydrogen at $T = 10,000$, $25,000$, and $50,000$ K. (b) The distribution of $v_\parallel$ for hydrogen, carbon, magnesium, and iron at $T = 50,000$ K. (c) The distribution of particle speeds, $v$, for hydrogen at $T = 10,000$, $25,000$, and $50,000$ K. (d) The distribution of $v$ for hydrogen, carbon, magnesium, and iron at $T = 50,000$ K.

for electrons $m = m_e$, $T = T_e$ assuming electrons are thermelized.

for atomic species $k$, $m = m_k$, $T = T_k$ kinetic temperature of species $k$.

If electrons and nuclear particles are thermelized, then particles are in equilibrium, gas has single temperature.

- however, for thermal equilibrium, all particles must be thermelized at a given $T$ and radiation must be thermelized at this same $T$. If so, then radiation distribution function is a Planck function.
Radiation Field

- In thermal equilibrium, the radiation field is the Planck function

\[ B_\nu(T) = \frac{2\pi h\nu^3}{c^2} \frac{1}{\exp(h\nu/kT) - 1} \text{ erg s}^{-1} \text{ cm}^{-2} \text{ Hz}^{-1} \text{ Sr}^{-1} \]

- Provides the energy per second per unit area per unit of frequency per unit of solid angle. (into unit solid angle)

- In wavelength units

\[ B_\lambda(T) = \frac{2\pi c^2}{\lambda^5} \frac{1}{\exp(hc/\lambda kT) - 1} \text{ erg s}^{-1} \text{ cm}^{-2} \text{ A}^{-1} \text{ Sr}^{-1} \]

per unit angstrom

- The wavelength of peak is \( \lambda_{\text{max}} = \frac{5.1 \times 10^{-7}}{T} \text{ A} \)

- Integrating over all frequency

\[ B(T) = \int_0^\infty B_\nu(T) d\nu = \frac{\sigma}{\pi} T^4 \]

\[ \text{Flux} = \pi B(T) = \sigma T^4 \]

- Recall that Flux = \( \frac{L}{4\pi R_x^2} = \sigma T^4 \)

\[ \sigma = \text{Stefan-Boltzmann constant} = \frac{2\pi^5 K}{15h^3 c^2} = 5.67 \times 10^{-5} \quad \text{erg s}^{-1} \text{ cm}^{-2} \text{ K}^{-4} \]

- Thus, the radiation field in a gas in thermal equilibrium is dependent upon one physical macroscopic parameter = T
Fig. 1. The brightness, $B_v(T)$, of a black-body radiator at frequency, $v$, and temperature, $T$. The Planck function, $B_v(T)$, is given by Eq. (1-119)

$$E = \sigma T^4$$ energy under curve

Fig. 1-1 Wavelength Distributions of Black Body Radiation for Various Temperatures. The temperatures 12,000°K, 9000°K, 6000°K, and 3000°K are in the ratio 4:3:2:1. Hence the wavelengths of maximum intensity, according to Wien’s law, are in the ratio 1:2:3:4, respectively, and are 0.2415 $\mu$, 0.3220 $\mu$, 0.4830 $\mu$, and 0.9659 $\mu$. The areas under the curves, as given by the Stefan-Boltzmann law, are $4^4:3^4:2^4:1$ or 256:81:16:1. The two short strokes below the horizontal axis indicate the approximate range of visible radiation. This encompasses the most intense part of the emission at 6000°K, which corresponds closely to the wavelength distribution of the energy emitted by the Sun (cf. Fig. 1-2). (Novotny)
Mean Kinetic Energy

Consider particles of mass $m$ in non-relativistic gas

$$\langle KE \rangle = \frac{1}{2} m \langle v^2 \rangle$$

where

$$\langle v^2 \rangle = \frac{\int_0^\infty v^2 f(v) \, dv}{\int_0^\infty f(v) \, dv} = 1$$

$f(v)$ = Maxwellian speed distribution

$$f(v) \, dv = \left( \frac{m}{2\pi kT} \right)^{\frac{3}{2}} \exp\left\{ - \frac{m v^2}{2kT} \right\} \frac{v^2}{2} \, dv$$

We obtain

$$\langle v^2 \rangle = \int_0^\infty v^2 f(v) \, dv = \frac{4}{11} \frac{1}{v_0^3} \int_0^\infty \exp\left\{ - \frac{v^2}{v_0^2} \right\} v^4 \, dv$$

where $v_0 = \left( \frac{2kT}{m} \right)^{\frac{1}{2}}$ = most probable speed

and where

$$\int_0^\infty \exp\left\{ - \frac{v^2}{v_0^2} \right\} v^4 \, dv = \frac{3}{8} \pi \frac{1}{2} v_0^5$$

$$\langle v^2 \rangle = \left( \frac{4}{11} \frac{1}{v_0^3} \right) \left( \frac{3}{8} \pi \frac{1}{2} v_0^5 \right) = \frac{3}{2} \frac{v_0^2}{v_0^2} = \frac{3}{2} \frac{kT}{m}$$

Substituting into $\langle KE \rangle$:

$$\langle KE \rangle = \frac{1}{2} m \langle v^2 \rangle = \frac{3}{2} \frac{kT}{m}$$

per particle. $T = T_k$ is clearly the "kinetic" temperature.

**NOTE:** Average $\langle KE \rangle$ is independent of particle mass — thus, if all particles are thermalized and have some $T$, then the mean kinetic energy of the gas is simply

$$\frac{3}{2} n k T$$

$n$ = total number density
Particle Pressure (and definition of mean molecular weight)

Pressure = Force per unit area = \( \frac{1}{A} \frac{dp}{dt} \)

\( p = \text{momentum of particles} \)

\[
\begin{align*}
\text{momentum per particle of mass } m &= mv \\
\text{Pressure} &= (mv) \frac{dn}{dt}
\end{align*}
\]

where \( \frac{dn}{dt} = \text{rate at which number density of particles pass through unit area in time dt} \)

\[
\frac{dn}{dt} = \frac{1}{3} n v
\]

\[ \{ \text{rate at which particles pass through unit area on average} \} \]

thus

\[
\langle P \rangle = \langle mv, \frac{1}{3} n v \rangle = \frac{1}{3} m n \langle v^2 \rangle = \frac{p}{3} \langle v^2 \rangle
\]

from \( \langle KE \rangle \) we found \( \langle v^2 \rangle = 3 \frac{kT}{m} \)

thus

\[
\langle P \rangle = \frac{p}{m} kT
\]

since \( \langle KE \rangle = \frac{3}{2} kT \) we see that

\[
\langle P \rangle = \frac{p}{m} \left( \frac{3}{8} \langle KE \rangle \right)
\]

Mean Molecular Weight

*This mean particle pressure applies to a single class of particle of mass \( m \).

In actuality we must account for the mean mass of the gas, such that

\[
P_{\text{av}} = \frac{p}{\langle m \rangle} kT
\]

where

\[
\langle m \rangle = \mu m_{\text{amu}}
\]

\[
m_{\text{amu}} = 1.66053878 \times 10^{-24} \text{ g}
\]

\( \mu \equiv \text{mean molecular weight} = \text{mean mass of all particles in amu units}. \)

Yielding

(for mixture of particles)

\[
\langle P \rangle = \frac{p}{\mu m_{\text{amu}}} kT
\]

\{ sometimes authors employ \}

\[
M_{\text{amu}} = M_H
\]

\{ technically incorrect \}

\( \text{even though } m_{\text{amu}} \neq M_H \)
Total Pressure in a gas is sum of partial pressures

\[ P = \sum P_i \]

Main pressures:

\( P_N \) = pressure from atom/nuclear particles including bare nuclei, partially ionized atoms, neutral atoms.

\( P_e \) = electron pressure from free electrons in the gas.

\( P_{rad} \) = radiation pressure due to photons

\[ P = P_N + P_e + P_{rad} \]

define \( P_{gas} = P_N + P_e \)

\[ P = P_{gas} + P_{rad} \]

\[ P_N = \frac{N_N \cdot kT}{N_{N_{max}}} \]

\[ P_e = \frac{N_e \cdot kT}{\mu_e m_{e0}} \]

Note:

\[ n = \frac{N_N}{N_{N_{max}}} \]

or \( P = N_{N_{max}} n \)

\( P_{rad} = \frac{1}{3} U = \frac{2}{3} \cdot a \cdot T^4 \)

\( U = \text{energy density of photons} \quad [\text{erg cm}^{-3}] \)

\( a = \text{radiation density constant} \)

\( c = 7.56 \times 10^{-15} \quad [\text{erg cm}^{-3} \text{ K}^{-4}] \)
number densities, mass densities, mass fractions

\[ n_i = \text{number of particles per unit volume } [\text{cm}^{-3}] \]
\[ \rho_i = \text{mass of particles per unit volume } [\text{g cm}^{-3}] \]
\[ x_i = \text{fraction of } \rho_i \text{ to total } \rho = \sum \rho_i \text{ mass fraction [unitless]} \]
\[ \uparrow x_i = \frac{\rho_i}{\sum \rho_i} \]

for nuclear/atomic particles, define subscripts:

- \( i \) = excitation level \( i = 1, 2, \ldots \infty \)
- \( j \) = ionization level \( j = 1 \) neutral \( j = 1, 2, 3, \ldots Z+1 \)
- \( k \) = species \( k = 1 \) hydrogen, \( k = 2 \) helium, etc.

use convention that \( k \) = atomic number

then \( j \) ranges from 1 to \( k+1 \)

we then have:

\[ n_{ijk} = \text{number density of species } k \text{ in ionization stage } j \text{ and excitation state } i \]

then

\[ n_{jk} = \sum_i n_{ijk} \text{ number density of species } k \text{ in ionization stage } j \text{ (includes all excitation states)} \]

\[ n_k = \sum_{j=1}^{k+1} n_{jk} = \sum_{j=1}^{k+1} \sum_i n_{ijk} \text{ number density of species } k \text{ (includes all ionization states in all excitation states)} \]
Abundances

- three ways to measure $\xi_k$

1. $\tilde{\xi}_k = \log \left( \frac{n_k}{n_H} \right) + 12.00$ commonly tabulated (Asplund, Grevesse, $+$ Sauval 2005, astro-ph/0410214)

2. $\frac{n_k}{n_H} = \text{density wrt to hydrogen} \rightarrow \text{just solve from } \tilde{\xi}_k$

3. $\xi_k = \frac{n_k}{n_N}$, $n_N = \sum_k n_k$

where $\sum_k \xi_k = 1$

- defining metallicity/abundances wrt solar

\[ \left[ \frac{X}{H} \right] = \log \left( \frac{X}{H} \right) - \log \left( \frac{X}{H} \right)_0 = 0 \text{ for solar abundance of } X \]

$\frac{X}{H} = \frac{n_k}{n_H}$ of target object

$\left( \frac{X}{H} \right)_0 = \frac{n_k}{n_H}$ of solar photosphere (sometimes including meteoric values)

ex. \[ \left[ \frac{X}{H} \right] = -0.5 \text{ is } \frac{1}{2} \text{ dex below solar abundance} \]
Table 1. Element abundances in the present-day solar photosphere and in meteorites (Cl chondrites). Indirect solar estimates are marked with [..]

<table>
<thead>
<tr>
<th>Elem.</th>
<th>Photosphere</th>
<th>Meteorites</th>
<th>Elem.</th>
<th>Photosphere</th>
<th>Meteorites</th>
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<tbody>
<tr>
<td>1 H</td>
<td>12.00</td>
<td>8.25 ± 0.05</td>
<td>44 Ru</td>
<td>1.84 ± 0.07</td>
<td>1.77 ± 0.08</td>
</tr>
<tr>
<td>2 He</td>
<td>[10.93 ± 0.01]</td>
<td>1.29</td>
<td>45 Rh</td>
<td>1.12 ± 0.12</td>
<td>1.07 ± 0.02</td>
</tr>
<tr>
<td>3 Li</td>
<td>1.05 ± 0.10</td>
<td>3.25 ± 0.06</td>
<td>46 Pd</td>
<td>1.69 ± 0.04</td>
<td>1.67 ± 0.02</td>
</tr>
<tr>
<td>4 Be</td>
<td>1.38 ± 0.09</td>
<td>1.38 ± 0.08</td>
<td>47 Ag</td>
<td>0.94 ± 0.24</td>
<td>1.20 ± 0.06</td>
</tr>
<tr>
<td>5 B</td>
<td>2.70 ± 0.20</td>
<td>2.75 ± 0.04</td>
<td>48 Cd</td>
<td>1.77 ± 0.11</td>
<td>1.71 ± 0.03</td>
</tr>
<tr>
<td>6 C</td>
<td>8.39 ± 0.05</td>
<td>7.40 ± 0.06</td>
<td>49 In</td>
<td>1.60 ± 0.20</td>
<td>0.80 ± 0.03</td>
</tr>
<tr>
<td>7 N</td>
<td>7.78 ± 0.06</td>
<td>6.25 ± 0.07</td>
<td>50 Sn</td>
<td>2.00 ± 0.30</td>
<td>2.08 ± 0.04</td>
</tr>
<tr>
<td>8 O</td>
<td>8.66 ± 0.05</td>
<td>8.39 ± 0.02</td>
<td>51 Sb</td>
<td>1.00 ± 0.30</td>
<td>1.03 ± 0.07</td>
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<tr>
<td>9 F</td>
<td>4.56 ± 0.30</td>
<td>4.43 ± 0.06</td>
<td>52 Te</td>
<td>2.19 ± 0.04</td>
<td></td>
</tr>
<tr>
<td>10 Ne</td>
<td>[7.84 ± 0.06]</td>
<td>-1.06</td>
<td>53 I</td>
<td></td>
<td>1.51 ± 0.12</td>
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<td>11 Na</td>
<td>6.17 ± 0.04</td>
<td>6.27 ± 0.03</td>
<td>54 Xe</td>
<td>[2.27 ± 0.02]</td>
<td>-1.97</td>
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<tr>
<td>12 Mg</td>
<td>7.53 ± 0.09</td>
<td>7.53 ± 0.03</td>
<td>55 Cs</td>
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<td>1.07 ± 0.03</td>
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<tr>
<td>13 Al</td>
<td>6.37 ± 0.06</td>
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<td>56 Ba</td>
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<td>7.51 ± 0.02</td>
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<td>5.40 ± 0.04</td>
<td>58 Ce</td>
<td>1.58 ± 0.09</td>
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<td>7.16 ± 0.04</td>
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<td>0.75 ± 0.03</td>
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<td>5.23 ± 0.06</td>
<td>60 Nd</td>
<td>1.45 ± 0.05</td>
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<td>-0.45</td>
<td>62 Sm</td>
<td>1.01 ± 0.06</td>
<td>0.92 ± 0.04</td>
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<td>19 K</td>
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<td>5.06 ± 0.05</td>
<td>63 Eu</td>
<td>0.52 ± 0.06</td>
<td>0.49 ± 0.04</td>
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<td>20 Ca</td>
<td>6.31 ± 0.04</td>
<td>6.29 ± 0.03</td>
<td>64 Gd</td>
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<td>1.03 ± 0.02</td>
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<td>3.04 ± 0.04</td>
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<td>4.89 ± 0.03</td>
<td>66 Dy</td>
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<td>1.10 ± 0.04</td>
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<td>3.97 ± 0.03</td>
<td>67 Ho</td>
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<td>0.46 ± 0.02</td>
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<td>5.64 ± 0.10</td>
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<td>68 Er</td>
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<td>0.92 ± 0.03</td>
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<td>69 Tm</td>
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<td>80 Hg</td>
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<td>2.92 ± 0.05</td>
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<td>82 Pb</td>
<td>2.00 ± 0.06</td>
<td>2.02 ± 0.04</td>
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<tr>
<td>39 Y</td>
<td>2.21 ± 0.02</td>
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<td>83 Bi</td>
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<td>0.65 ± 0.03</td>
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<td>40 Zr</td>
<td>2.59 ± 0.04</td>
<td>2.57 ± 0.02</td>
<td>90 Th</td>
<td></td>
<td>0.06 ± 0.04</td>
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<td>41 Nb</td>
<td>1.42 ± 0.06</td>
<td>1.39 ± 0.03</td>
<td>92 U</td>
<td>&lt;0.47</td>
<td>-0.52 ± 0.04</td>
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<tr>
<td>42 Mo</td>
<td>1.92 ± 0.05</td>
<td>1.96 ± 0.04</td>
<td></td>
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</table>
Obtaining $\xi_k$ from $\hat{\alpha}_k$

$$\xi_k = \frac{n_k}{n_W}, \quad n_W = \sum n_k$$

In terms of $\hat{\alpha}_k$

$$\alpha_k = \frac{n_k/n_W}{n_{\hat{\alpha}}/n_H} \quad \text{where} \quad \frac{n_k}{n_W} = 10^{(\hat{\alpha}_k - 12)} \quad \text{and} \quad \frac{n_W}{n_H} = \sum_{k} 10^{(\hat{\alpha}_k - 12)}$$

yielding

$$\alpha_k = \frac{10^{(\hat{\alpha}_k - 12)}}{\sum_{k} 10^{(\hat{\alpha}_k - 12)}}$$

Mass Fractions

$$x_k = \frac{p_k}{p_{\hat{\alpha}}}, \quad n_k = \sum x_k$$

$X$ = mass fraction of hydrogen

$Y$ = mass fraction of helium

$Z$ = mass fraction of metals

$$X = \frac{p_H}{\sum_{k} p_k}, \quad Y = \frac{p_{He}}{\sum_{k} p_k}, \quad Z = \frac{\sum_{k} p_k}{\sum_{k} x_k}$$

Relation between $X$, $Y$, $Z$ and $\xi_H$, $\xi_{He}$, etc.

$$X = \frac{p_H}{\sum_{k} p_k} = \frac{n_H A_H \mathcal{M}_{env}}{\sum_{k} n_k A_k \mathcal{M}_{env}} = \frac{A_H n_H}{\sum_{k} A_k x_k} = \frac{A_H \xi_H}{\sum_{k} A_k x_k} = \frac{A_H}{A_H} \xi_H$$

$$\xi_H = \left(\frac{X}{A_H}\right) \sum A_k x_k$$
Similarly, we have
\[
Y = \sum_{k=1}^{\infty} \frac{A_{\text{He}} \alpha_k}{A_{\text{He}} \alpha_k} \alpha_k \approx \left( \frac{Y}{A_{\text{He}}} \right) \sum_{k=1}^{\infty} \frac{A_k \alpha_k}{A_{\text{He}} \alpha_k}
\]
and
\[
Z = \sum_{k=3}^{\infty} \frac{A_k \alpha_k}{A_{\text{He}} \alpha_k} \approx \left( \frac{Z}{A_{\text{He}}} \right) \sum_{k=3}^{\infty} \frac{A_k \alpha_k}{A_{\text{He}} \alpha_k}
\]

\[\alpha_k = \text{cannot be solved in general because more than 1 metal!} \]

(see below)

with the constraint \[x + y + z = 1 \quad \text{so} \quad z = 1 - x - y\]

Solar values of mass fractions

\[
\begin{align*}
x_\odot & = 0.70683 \pm 0.018 \\
y_\odot & = 0.27431 \pm 0.016 \\
z_\odot & = 0.0188 \pm 0.0016
\end{align*}
\]

\[\begin{array}{c}
\text{commonly approx values} \\
X = 0.70 \quad Y = 0.28 \quad Z = 0.02 \\
X = 0.71 \quad Y = 0.27 \quad Z = 0.02
\end{array}\]

\[\alpha_{\text{single metal}} = \alpha_3 = \left( \frac{Z}{A_3} \right) \sum_{k=1}^{\infty} \frac{A_k \alpha_k}{A_{\text{He}} \alpha_k} \quad k = 1, 2, 3
\]

from \[\alpha_\text{H} + \alpha_{\text{He}} + \alpha_3 = 1\]
we have

\[
\alpha_\text{H} \left[ 1 + \frac{\alpha_{\text{He}} + \alpha_3}{\alpha_\text{H}} \right] = 1
\]

\[\alpha_3 = \left[ 1 + \frac{\alpha_{\text{He}} + \alpha_3}{\alpha_\text{H}} \right]^{-1}
\]

\[\frac{\alpha_{\text{He}}}{\alpha_\text{H}} = \frac{A_{\text{He}}}{A_{\text{He}}} \left( \frac{Y}{X} \right)
\]

\[\frac{\alpha_3}{\alpha_\text{H}} = \frac{A_3}{A_3} \left( \frac{Z}{X} \right)
\]

\[\alpha_3 = \alpha_\text{H} \frac{A_3}{A_3} \left( \frac{Z}{X} \right), \quad \alpha_{\text{He}} = \alpha_\text{H} \frac{A_{\text{He}}}{A_{\text{He}}} \left( \frac{Y}{X} \right), \quad \text{where} \quad \alpha_\text{H} = \left[ 1 + \frac{A_\text{H}}{A_{\text{He}}} \left( \frac{Y}{X} \right) + \frac{A_3}{A_3} \left( \frac{Z}{X} \right) \right]^{-1}
\]
USEFUL RELATIONS

\[ n_N = \sum n_k \]

number density of all nuclear/atomic particles

the same applies for mass densities

\[ \rho_{jk} = \sum \rho_{ij} \]
\[ \rho_k = \sum \rho_{ik} \]
\[ \rho = \sum \rho_k \]

- mass fractions defined

\[ x_k = \frac{\rho_k}{\rho_n} = \frac{\rho_{ik}}{\rho_{rk}} \]

- relationships between \( \rho_k, M_k, x_k \) for given atom species (not mixture of species)

\[ \rho_k = M_k n_k \]
\[ M_k = M_{amu} A_k \]
\[ \rho_k = n_k M_{amu} A_k \]

invoking \( x_k = \frac{\rho_k}{\rho_n} \)

\[ n_k = \frac{x_k \rho_n}{A_k M_{amu}} \]

or

\[ \rho_n = \frac{M_{amu} A_k}{x_k n_k} \]

- these are useful relationships and will be invoked often
- identical equations apply for electrons \( e, M_e, x_e \)
Particle and Charge Conservation - Solving for $n_e$

- for a non-relativistic gas

$$P_{\text{gas}} = n_k kT = (n_N + n_e) kT$$

where

$$n_k = \sum_{j=1}^{K+1} \sum_{i=1}^{J} \frac{n_{jk}}{n_k}$$

as such, we can write

$$P_{\text{gas}} = P_N + P_e = n_k kT + n_e kT$$

- the density of electrons is given by

$$n_e = \sum_{j=1}^{K+1} \sum_{i=1}^{J} n_{jk} / n_k$$

from

$$\frac{n_{jk}}{n_k} = f_{jk}(n_j, T)$$

- electron density is

$$n_e = \sum_{j=1}^{K+1} n_{jk} \sum_{i=1}^{J} (j-1) P_{jk}$$

- use $n_N = n - n_e$ and $\alpha_k = n_k / n_N = n_k / (n_N + n_e)$

yielding the transcendental equation

$$n_e = (n - n_e) \left\{ \sum_{k=1}^{K+1} \sum_{i=1}^{J} \alpha_k (j-1) f_{jk}(n_j, T) \right\}$$

- the total gas pressure from particles is sum of atoms and ions (denoted $n_N$) and electrons (denoted $n_e$)

- sum of all neutrals and ionized species

- $n = n_N + n_e$ with subscript denotes total

- sum of $j$ provides electron density contributed by ion $j$ of species $k$, i.e. $n_{jk}$; over all $k$ ionization levels

- ionization fraction

- use if you know all $n_k, f_{jk}$ - the problem is that the $f_{jk}$ depend on $n_e$; so how to obtain $n_e$?

- $\alpha_k$ is abundance fraction

So, if you know $n$ and $\alpha_k$, you must solve this for $n_e$
Computing mean molecular weights.

Recall
\[ P = n k T = \frac{k}{\mu M_{\text{amu}}} \rho T \]
\[ \Rightarrow n = \frac{\rho}{\mu M_{\text{amu}}} \]

Three types of \( n \):

1. \( n_U \) = nuclear particles.
2. \( n_e \) = electrons
3. \( n \) = total gas

1) Nuclear particles.
\[ P_n = \frac{k}{\mu M_{\text{amu}}} e_n T \]
\[ n_n = \frac{e_n}{\mu M_{\text{amu}}} \]

Useful relations
\[ n_k = \frac{e_k}{m_k} \] number density of species \( k \) \( [\text{cm}^{-3}] \)

\[ m_k = M_{\text{amu}} A_k \] mass \( [\text{g}] \) of species \( k \), where \( A_k \) = atomic mass

\[ X_k = \frac{P_k}{P_n} \] mass fraction of species \( k \)

\[ n_k = \frac{X_k}{A_k} \frac{e_n}{M_{\text{amu}}} \]

For all nuclear/atomic species
\[ n_n = \sum_k n_k = \frac{e_n}{M_{\text{amu}}} \sum_k \left( \frac{X_k}{A_k} \right) \]

Applying \( n = \frac{\rho}{\mu M_{\text{amu}}} \), we have \( \frac{\rho_n}{\mu M_{\text{amu}}} = \frac{e_n}{M_{\text{amu}}} \sum_k \left( \frac{X_k}{A_k} \right) \)

Yielding
\[ n_n = \left\{ \sum_k \left( \frac{X_k}{A_k} \right) \right\}^{-1} \]
[2] electrons \[ \rho_e = \frac{k}{N_e m_e u} \rho_e T \]
\[ n_e = \frac{\rho_e}{N_e m_e u} \]

Electron density is contributed by ions, a single ion of species \( k \) in ionization stage \( j \) contributes \((j-1)n_{jk}\) to \( n_e \). Thus, the contribution from species \( k \) including all ionization stages is \( \sum_{j=1}^{k+1} (j-1)n_{jk} \). The contribution from all ions of all species is then

\[ n_e = \sum_{k=1}^{k+1} \sum_{j=1}^{k+1} (j-1)n_{jk} \]

Invoking the ionization fraction, \( n_{jk} = f_{jk} n_k \),

we have

\[ n_e = \sum_{k=1}^{k+1} \sum_{j=1}^{k+1} (j-1)f_{jk} \]

From \( n_k = \frac{x_k}{A_k m_e u} \), we have

\[ n_e = \frac{\rho_n}{\rho_n m_e u} \left\{ \sum_{k=1}^{k+1} \sum_{j=1}^{k+1} \frac{x_k}{A_k} (j-1)f_{jk} \right\} \]

Note that \( n_e \) is expressed in terms of \( \rho_n \! \).}

From \( n_e = \frac{\rho_e}{N_e m_e u} \), we have

\[ n_e = \frac{\rho_e}{\rho_n} \left\{ \frac{x_k}{A_k} \sum_{j=1}^{k+1} (j-1)f_{jk} \right\}^{-1} \]
WE USE THE CONVENTION THAT TOTAL GAS TERMS HAVE NO SUBSCRIPT

\[ n = n_n + n_e = \frac{P}{N_{m_{00}}} \]

RECALL

\[ n_n = \frac{P_n}{N_n m_{00}} = \frac{P_n}{m_{00}} \left[ \sum \frac{X_k}{A_k} \right] \rightarrow N_n = \left\{ \sum \frac{(X_k)}{A_k} \right\}^{-1} \]

\[ n_e = \frac{P_e}{N_e m_{00}} = \frac{P_n}{m_{00}} \left[ \sum \frac{X_k}{A_k} \sum (j-1) P_{jk} \right] \rightarrow N_e = \frac{P_e}{P_n} \left\{ \sum \frac{X_k}{A_k} \sum (j-1) P_{jk} \right\}^{-1} \]

\[ n = n_n + n_e = \frac{P}{N_{m_{00}}} = \frac{P_n}{m_{00}} \left\{ \sum \frac{X_k}{A_k} \sum (j-1) P_{jk} \right\} \]

\[ \frac{P}{N_{m_{00}}} = \frac{P_n}{m_{00}} \left[ \frac{1}{N_n} + \frac{1}{N_e} \right] \]

\[ \mu = \frac{P}{P_n} \left[ \frac{1}{N_n} + \frac{1}{N_e} \right]^{-1} \quad \text{w/} \quad N_e' = \left\{ \sum \frac{X_k}{A_k} \sum (j-1) P_{jk} \right\}^{-1} \]

INTERESTINGLY, THIS YIELDS

\[ P_g = \frac{K}{m_{00}} \left[ \frac{1}{N_n + N_e'} \right] P_n T \]

TOTAL GAS PRESSURE IN TERMS OF \(P_n\)!

\[ \rightarrow \text{to a good approximation, this can be simplified} \rightarrow \]
The massless electron approximation

consider a fully ionized gas (where \( n_e \) is its maximum)
in this case all \( f_{jk} = 0 \) except \( f_{kk} = 1 \), thus

\[
\rho_e = m_e n_e = \frac{\rho_{n}}{M_{nu}} \left\{ \sum_{k} \frac{x_k}{P_{ke}} \sum_{j=1}^{k+1} f_{jk} \right\} = m_e \frac{\rho_{n}}{M_{nu}} \sum_{k} \frac{x_k}{P_{ke}} n_k
\]

from \( n_k = \frac{\rho_{n}}{M_{nu}} \frac{x_k}{P_{ke}} \) we have

\[
\rho_e = m_e \sum_{k} x_k n_k
\]

for nuclear particles

\[
\rho_{n} = \sum_{k} m_k n_k = \frac{M_{nu}}{M_{nu}} \sum_{k} A_k n_k
\]

for all species except hydrogen \( A_k \approx 2k \), \( \rho_{n} = 2 \frac{M_{nu}}{M_{nu}} \sum_{k} x_k n_k \)

we show that \( \rho = \rho_{n} \)

\[
\rho = \rho_{n} + \rho_e = \rho_{n} \left( 1 + \frac{\rho_e}{\rho_{n}} \right) = \rho_{n} \left[ 1 + \frac{m_e \sum_{k} x_k n_k}{2 \frac{M_{nu}}{M_{nu}} \sum_{k} A_k n_k} \right] = \rho_{n} \left[ 1 + \frac{m_e}{2 \frac{M_{nu}}{M_{nu}}} \right]
\]

\[
\frac{m_e}{2 \frac{M_{nu}}{M_{nu}}} = \frac{9.1 \times 10^{-28}}{2 \times (1.6 \times 10^{-24})} = 2.8 \times 10^{-4}
\]

\( \rho = \rho_{n} \) even in fully ionized gas

\[
\sigma \approx \rho_{n}
\]

\[
N = \frac{\rho}{\rho_{n}} \left[ \frac{1}{N_{n}} + \frac{1}{N_{e}} \right] = \frac{\rho_{n} + \rho_e}{\rho_{n}} \left[ \frac{1}{N_{n}} + \frac{1}{N_{e}} \right] = \left( 1 + \frac{\rho_e}{\rho_{n}} \right) \left[ \frac{1}{N_{n}} + \frac{1}{N_{e}} \right] = \left( 1 + \frac{\rho_{n}}{\rho_{n}} \right) \left[ \frac{1}{N_{n}} + \frac{1}{N_{e}} \right]
\]

giving (to excellent approx)

\[
N = \left[ \frac{1}{N_{n}} + \frac{1}{N_{e}} \right]
\]

with \( \rho_{g} = \frac{K}{M_{nu}} \rho \) not \( \rho_{n} \)
\[ N = \left[ \frac{1}{N_N} + \frac{1}{N_e} \right]^{-1} \]

\[ N_N = \left( \sum_{k=1}^\infty \frac{X_k}{A_k} \right)^{-1} = \left( \frac{X_H}{A_H} + \frac{X_{He}}{A_{He}} + \sum_{k=3}^\infty \frac{X_k}{A_k} \right)^{-1} = \left( X + \frac{1}{4} Y + \sum_{k=3}^\infty \frac{X_k}{A_k} \right)^{-1} \quad \text{where } A_{He} > 4 \]

\[ N_e = \left( \sum_{k=1}^\infty \frac{X_k}{A_k} \sum_{j=1}^{k-1} f_{jk} \right)^{-1} = \left( \frac{X_H}{A_H} (1) + \frac{X_{He}}{A_{He}} (2) + \sum_{k=3}^\infty \frac{X_k}{A_k} \right)^{-1} \]

\[ = \left( X + \frac{1}{2} Y + \sum_{k=3}^\infty \frac{X_k}{Z} \right)^{-1} \quad \text{where we invoke } A_k = 2k \text{ in the sum} \]

\[ N = \left[ \frac{1}{N_N} + \frac{1}{N_e} \right]^{-1} = \left[ \left( X + \frac{1}{4} Y + \sum_{k=3}^\infty \frac{X_k}{A_k} \right) + \left( X + \frac{1}{2} Y + \sum_{k=3}^\infty \frac{X_k}{Z} \right) \right]^{-1} \]

\[ = \left[ 2X + \frac{3}{4} Y + \sum_{k=3}^\infty X_k \left( \frac{2 + A_k}{Z A_k} \right) \right]^{-1} \]

as \( k \) becomes large \( \frac{2 + A_k}{Z A_k} \approx \frac{1}{Z} \) and sum becomes \( \frac{1}{Z} \sum_{k=2}^\infty \frac{X_k}{k} = \frac{1}{2} Z \)

thus

\[ N = \left[ 2X + \frac{3}{4} Y + \frac{1}{2} Z \right]^{-1} \]

using \( X + Y + Z = 1 \) \( Z = 1 - X - Y \), we also have

\[ N \approx 2 \left[ 3X + \frac{1}{2} Y + 1 \right]^{-1} \]