ASTR 545 Fall 2018
Homework 4 (100 points)
ALWAYS SHOW YOUR WORK

Consider four stars with solar abundances, which have the following properties at $\tau = 2/3$ in their atmospheres:
K5V: $T = 4470$ K ; $n_e = 2.6 \times 10^{14}$ cm^{-3}
G5V: $T = 5930$ K ; $n_e = 1.2 \times 10^{14}$ cm^{-3}
A5V: $T = 8430$ K ; $n_e = 3.9 \times 10^{14}$ cm^{-3}
B5V: $T = 13,980$ K ; $n_e = 4.4 \times 10^{13}$ cm^{-3}

1. (35 pts) Neutral Hydrogen Bound-Free Absorption: In class, we discussed an accurate, yet approximated, expression for the neutral hydrogen bound-free absorption cross section and obtained

$$\alpha^{\rm bf}_{\rm HI}(\lambda) = A_0'\lambda^3 \left[\sum_{n=1}^{\infty} \frac{g_{\rm II}(n,\lambda)}{n^3} \exp\left\{-\frac{\chi_n}{kT}\right\}\right] \quad \text{cm}^2 \text{ per neutral H atom}\,, \tag{1}$$

where $A_0' = 1.0449 \times 10^{-26}$ cm^2 Å^{-3}, $g_{\rm II}(n,\lambda)$ is the bound-free Gaunt factor for level n

$$g_{\rm II}(n,\lambda) \simeq 1 - \left[\frac{\frac{121}{700}(1-\epsilon_n^2)}{n^{2/3}(1+\epsilon_n^2)^{2/3}}\right] , \quad \text{where} \quad \epsilon_n = \frac{\lambda_n}{\lambda} - 1 \quad \text{for} \quad 0 \leq \epsilon_n \leq 1 \tag{2}$$

and

$$g_{\rm II}(n,\lambda) \simeq 1 + \left[\frac{\frac{121}{700}(1-1/\epsilon_n^2)}{\kappa_n^{2/3}(1+1/\epsilon_n^2)^{2/3}}\right] , \quad \text{where} \quad \kappa_n = n\epsilon_n^{-1/2} \quad \text{for} \quad \epsilon_n > 1 \tag{3}$$

and $g_{\rm II}(n,\lambda) = 0$ otherwise, where

$$\lambda_n = n^2 \frac{hc}{R_{\rm H}} \quad \text{and} \quad \chi_n = R_{\rm H}\left(1 - \frac{1}{n^2}\right) , \tag{4}$$

where $R_{\rm H} = 13.598$ eV and χ_n is the excitation energy for level n.

(a) [5 pts] Given that $\alpha^{\rm bf}_n(\lambda)$, the cross section per neutral hydrogen atom in excitation state n, scales as n^{-5}, explain why $\alpha^{\rm bf}_{\rm HI}(\lambda)$, the cross section per neutral hydrogen atom, scales as n^{-3}.

As mentioned, Kramer's semi-classical cross section is weighted by the principle qunatum number as $\alpha^{\rm bf}_n(\lambda) \propto 1/n^5$. This is the cross section for ionization from excitation state n in units cm^2 per H I atom in excitation state n. To obtain the total cross section at a given wavelength requires the density weighted sum of individual $\alpha^{\rm bf}_n(\lambda)$. Since the total cross section is in units cm^2 per H I atom (all excitation states inclusive), each cross section is weighted by the ratio, $n_{n,\rm HI}/n_{\rm HI}$ in the summation. This ratio is given by the Boltzmann Equation, which is proportional to the statistical weight for level n, in that $g_n = 2n^2$.

That is, even though the cross section scales as n^{-5}, decreasing rapdily as n increases, the number of possible states in level n that can be occupied per H I atom scales as n^2. Thus, the individual terms of the density weighted cross section scale as n^{-3}.

(b) [30 pts] Compute $\alpha^{\rm bf}_{\rm HI}(\lambda)$ at $\tau = 2/3$ for the above four stars over the wavelength range 1000–20,000 Å. Use the Unsöld approximation with $m = 6$. For each star, report your value of the "Unsöld contribution" and your values of $\alpha^{\rm bf}_{\rm HI}(\lambda)$ at λ_n for $n = 2, 3, 4$. (Problem 3 will ask you to plot your results.) Hand in your commented code.

See Table 1 (below) and Figure 1 (following page)

Table 1: Data For Problem 1(b)

G5V	λ_n [Å]	$\alpha^{\rm bf}_{\rm HI}$ [cm^2]	Unsöld [cm^2]
2	3649	1.241×10^{-25}	1.936×10^{-29}
3	8210	1.271×10^{-26}	2.206×10^{-28}
4	14596	1.120×10^{-26}	1.239×10^{-27}
K5V	λ_n	$\alpha^{\rm bf}_{\rm HI}$	Unsöld
2	3649	1.825×10^{-28}	3.605×10^{-33}
3	8210	5.257×10^{-30}	4.106×10^{-32}
4	14596	2.952×10^{-30}	2.307×10^{-31}
A5V	λ_n	$\alpha^{\rm bf}_{\rm HI}$	Unsöld
2	3649	4.707×10^{-23}	4.729×10^{-26}
3	8210	1.557×10^{-23}	5.387×10^{-25}
4	14596	2.035×10^{-23}	3.027×10^{-24}
B5V	λ_n	$\alpha^{\rm bf}_{\rm HI}$	Unsöld
2	3649	1.324×10^{-20}	7.343×10^{-23}
3	8210	1.321×10^{-20}	8.363×10^{-22}
4	14596	2.437×10^{-20}	4.699×10^{-21}

2. (30 pts) The H$^-$ Ion Bound-Free Absorption: In class, we showed that an accurate laboratory data-fitted expression for the H$^-$ ion bound-free cross section is

$$\alpha^{\rm bf}_{\rm H^-}(\lambda) = 10^{-18} \sum_{i=0}^{6} a_i \lambda^i \quad \text{cm}^2 \text{ per neutral H}^- \text{ ion}\,, \tag{5}$$

where λ is units of angströms, and the a_i are given in the class notes. Compute the H$^-$ bound-free cross section, $\alpha^{\rm bf}_{\rm H^-/HI}(\lambda)$, which is in units cm^2 per neutral hydrogen atom, at $\tau = 2/3$ for these four stars over the wavelength range 1000–16,000 Å. To obtain $\alpha^{\rm bf}_{\rm H^-/HI}(\lambda)$, we must multiply $\alpha^{\rm bf}_{\rm H^-}(\lambda)$ by the ratio $n_{\rm H^-}/n_{\rm HI} = n_e/\Phi(T)$, where

$$\Phi(T) = 9.66 \times 10^{15}\, T^{3/2} \exp\left\{-\frac{0.755}{kT}\right\} \quad \text{cm}^{-3}\,. \tag{6}$$

For each star, report your $\alpha^{\rm bf}_{\rm H^-/HI}(\lambda)$ at λ_n for $n = 2$ (at the Balmer decrement). (Problem 3 will ask you to plot your results.) Hand in your commented code (you might combine your code for Problems 1 and 2, just inform me if so).

Problem 2 Solution: See the below table and Figure 1. Note that in the K5V star, the H^- contribution dominates at the Balmer ionization edge. Thus, one does not expect to measure a Balmer break. In the G5V star, the H^- also dominates. However, in the A5V star, the ratio reverses, so that the H I ionization edge is the dominant source of the opacity, leading to a Balmer break. On the B5V star, the H^- contribution is five orders or magnitude below that of the Balmer ionization edge, but in this case the ionization of hydrogen is almost 100%, so there is no discernible contribution to the opacity by neutral hydrogen.

Star	λ_2 [Å]	$\alpha^{\rm bf}_{\rm H^-}$ [cm^2]	$n_{\rm H^-}/n_{\rm HI}$	$\alpha^{\rm bf}_{\rm H^-/HI}$ [cm^2]
K5V	3649	2.064×10^{-17}	6.394×10^{-7}	1.320×10^{-23}
G5V	3649	2.064×10^{-17}	1.192×10^{-7}	2.460×10^{-24}
A5V	3649	2.064×10^{-17}	1.475×10^{-7}	3.044×10^{-24}
B5V	3649	2.064×10^{-17}	5.157×10^{-9}	1.064×10^{-25}

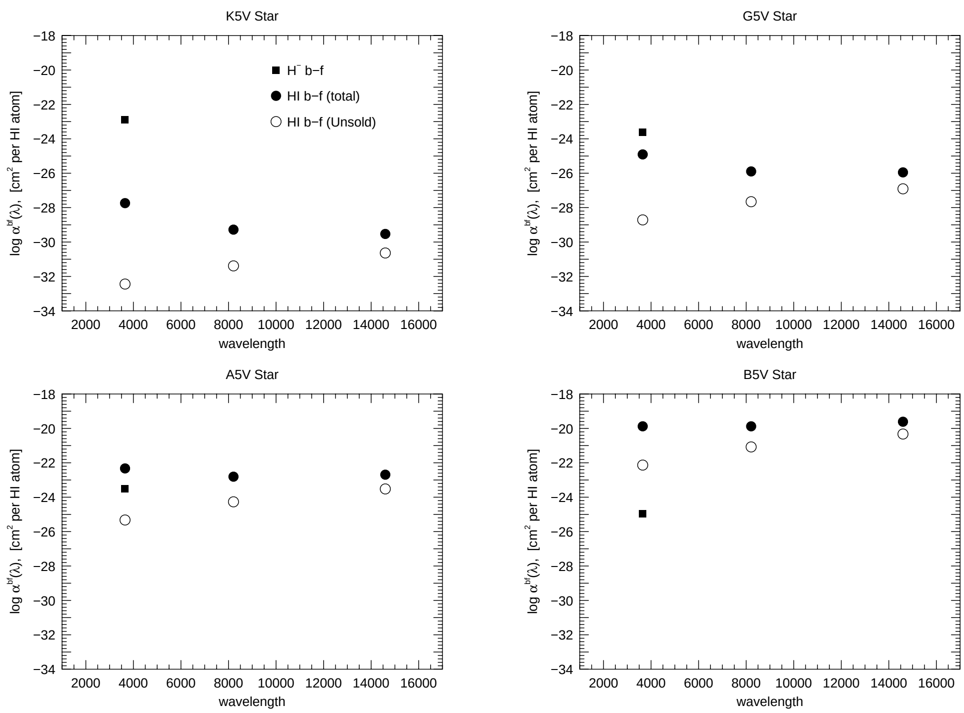

Figure 1: The bound-free (b-f) cross sections in units of cm^2 per H I atom plotted at the ionization edges for $n = 1, 2, 3$. Also shown are the Unsöld contributions to the H I b-f cross sections.

3. (35 pts) Plotting the "Total" Cross Section and Discussing the Balmer Decrement:

(a) [5 pts] For each star at $\tau = 2/3$, plot the ratio $n_{\mathrm{H}^-}/n_{\mathrm{HI}}$ (plot the curves on a single panel). Describe the trend you find from K5V to G5V to A5V to B5V stars.

> See Figure 2. As for temperature dependence, the ratio depends on $T^{-3/2}$, which you can see in the overall trend for the ratio to decrease as T increases. The ratio is linearly dependent on electron density, so it is proportional to $n_e T^{-3/2}$. The electron density will depend on the metallicity (which contributes greater numbers of free electrons from low ionization metals) and $\log(g)$ (as higher surface gravity stars have greater pressure and therefore higher densities). The electron densities I provided you do account for the general trend of n_e with $\log(g)$. Note the A5V star has the highest n_e, which I purposely chose to enhance (think of it as the A5V star having a higher metallicity). This raises the ratio proportionally.

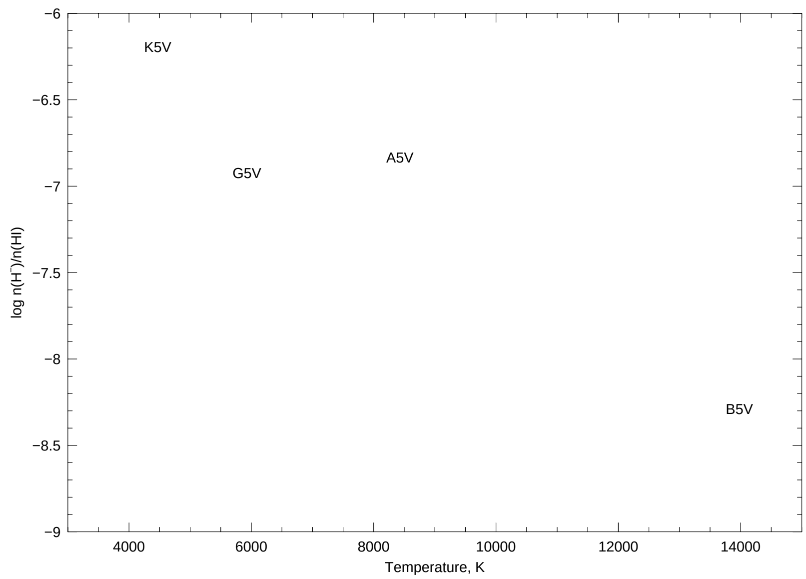

Figure 2: The ratio $n_{\mathrm{H}^-}/n_{\mathrm{HI}}$ for our four stars.

(b) [20 pts] For each star at $\tau = 2/3$, sum the cross sections to obtain a "total" bound-free cross section. For each star, plot your total cross section (solid curve) and your neutral hydrogen bound-free cross section (dashed or colored) and your H^- bound-free cross section (dotted or colored) over the wavelength range 1000-16,000 Å. Please present your plot of each star as a single panel and place all four panels on a single page. Provide a legend for the curves in at least one of the panels.

> See Figure 3. (next page)

(c) [5 pts] Based on your results, describe the anticipated behavior of the Balmer decrement as one goes from K5V to G5V to A5V to B5V stars. Reference the behavior of the total cross section with spectral class in your reasoning.

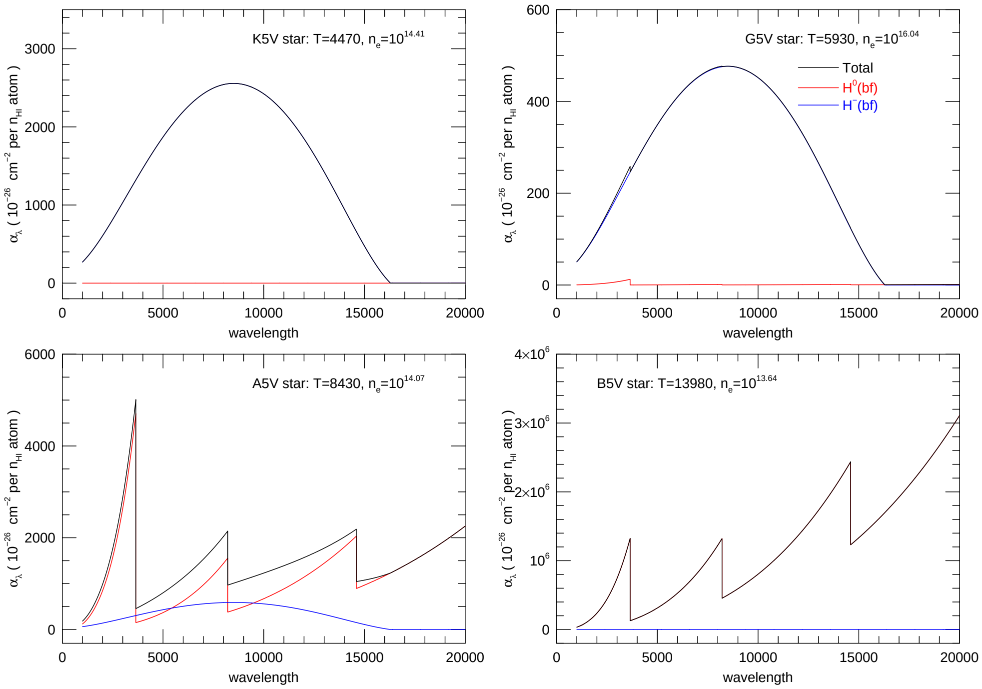

Figure 3: The bound-free cross continuous sections for our four stars.

It is clear that there is no Balmer decrement in the K5V star, as the continuous opacity, $\kappa_\lambda = n_{\rm HI}\alpha^{\rm bf}_{\rm HI}(\lambda)$ is entirely dominated by the H^- ion. The case is the same in the G5V star. However in the A5V star, we have that the opacity is dominated by the Balmer ionization edge. This is because the first excited state of H I is highly populated at this temperature (according to the Boltzmann Equation). The H^- ion contributes 5–10% to the opacity, and in fact dilutes the Balmer decrement in the spectrum because of the higher opacity just redward of the ionization edge (the black curve is higher than the red curve just redward of the break, but not discernibly so at the break). For the B5V star, the H I opacity total dominates, as H^- is virtually destroyed in this hot environment. However, there is no measurable Balmer break because κ_λ is vanishingly small, which occurs because $n_{\rm HI}$ is vanishingly small (the hydrogen is virtually all ionized).

(d) [5 pts] Consider the A5V star. If you increased the star's metallicity, then a greater number of electron donors (ionized metal atoms) would increase n_e in the gas. What affect would increasing the metallicity have on the observed size of the Balmer decrement and why?

Short answer... higher metallicity decreases the Balmer decrement (all other conditions being equal). Why? Consider the A5V star in Figure 3. Increasing the metallicity will increase the number density of electrons. As the ratio $n_{\rm H^-}/n_{\rm HI} \propto n_e$, we see that H^- opacity (the blue curve) would increase in amplitude. This would raise the opacity just redward of the Balmer ionization edge while not raising it discernibly at the edge. Not that changing the metallicity has no effect on the H I opacity.