ASTR 545
Grey Stellar Atmosphere Model

Compute a 1-D, plane-parallel, LTE, grey stellar atmosphere model over the optical depth range $10^{-3} \leq \tau \leq 10^{2}$. Solve a minimum of 10 equally spaced atmospheric layers per decade of optical depth (see me if you do not know how to divide equal space intervals in the log). Select a star as defined by its surface gravity, g, and effective temperature, $T_{\rm eff}$, and mass fractions X, Y, and Z (you are not required to use solar mass fractions). Use the OPAL opacities from the on-line Opacity Project (I will provide an electronic version of this table). Be mindful of the temperature range of this table; it applies only for hotter stars. If you wish to model a cooler star, you will need to supplement the OPAL opacity table using the Cox & Tabor (1976) table for lower temperatures (I will also provide this table). You must include at least one metal species, but are encouraged to incorporate as many as you can (for extra credit points).

Present your results in a "report" or "paper" written in Latex.

Include the following plots:

1. Fig. 1. Main Physical Quantities (four panels):
   (a) Physical Depth, x, versus optical depth, τ
   (b) Temperature, T, versus τ
   (c) Opacity, χ' [cm^2/g], versus τ
   (d) Total Mass Density, $\rho_{\rm tot}$, versus τ

2. Fig 2. Pressures (two panels):
   (a) P, P_g, $P_{\rm N}$, P_e, and P_r versus τ
   (b) P_e/P_g versus τ.

3. Fig 3. Ionization fractions (one panel):
   $f_{jk} = n_{jk}/n_k$ versus τ for all species

4. Fig 4. Free Electrons (one panel):
   Total electron density, n_e the partial electron densities (not fractions) from each n_{jk} versus τ.

5. Fig 5. Hydrogen excitation density fractions (one panel):
   n_{111}/n_1, n_{211}/n_1, and n_{311}/n_1 versus τ.

Provide a discussion for each figure and proper descriptive figure captions. Be mindful of when a log scale is appropriate and when a linear scale is appropriate. Be mindful of the limits of the axes so that the physically interesting information is cleary emphasized.

Also, include a table with the following columns: (1) optical depth, τ. (2) physical depth x ($x = 0$ at the "surface"), (3) temperature T, (4) total gas pressure P_g, (5) total mass density, ρ_{tot}, (6) electron pressure P_e, (7) electron density n_e, and (8) opacity τ, (9) the neutral hydrogen ionization fraction f_{11}.

In your write up please include the following:

1. Abstract: Begin by describing the star you modelled including the physical parameters defning your atmosphere. Describe the physical assumptions applied to the model. Describe and selected results of the model and provide a short interpretation and/or conclusion of the study.

2. Methods: Provide a detailed description of the mathematical model including physical assumptions (plane parallel, mean opacity and its definition, etc), equations used, and methods of computation, including the iterative scheme employed (to obtain pressure gradient), root-solving and interpolation schemes with references to pre-written codes, tables, or figures from books used (if any). After describing all the components, describe your algorithm for solution.

3. Results: Present your plots in order. If there are any stand-out features that you believe should be pointed out in the plot, then point them out and provide a brief interpretation.

4. Conclusion: This is a brief section in which you restate what you did and what you found (or discussed). You might end with a short blurb on what improvements could be made and implemented in the future to increase the realism of your model.

5. References: In journal style, provide the references for all work cited.

6. Code: provide your code.

GRADING: I will be looking for a clearly presented and well articulated results. Of course obtaining a physically sound model will be highly important, but if one does not complete the project, the presentsation of where the physical trouble lies, why you know the model is behaving non-physically, and what clue it may provide for resolving the problem, will be important to include. I will provide an additional 5 points per metal species included, up to 20 points additional. I will provide 5 points additional if one includes the Schwarzchild criteroin and reports which layers are unstable to convention (add a column to your table and discuss in your write up).

Upload your paper and your code to Canvas by the due date.

PARTICLE/MASS DENSITY CONSERVATION

$$n = n_{\rm N} + n_e\,, \quad \rho = \rho_{\rm N} + \rho_e$$

$$n_{\rm N} = \sum_k n_k\,, \quad n_k = \alpha_k n_{\rm N} = \sum_{j=1}^{k+1} n_{jk}\,, \quad n_{jk} = f_{jk} n_k = \sum_i n_{ijk}$$

$$\rho_{\rm N} = m_{\rm a} \sum_k A_k n_k = m_{\rm a} n_{\rm N} \sum_k \alpha_k A_k\,, \quad \rho_e = m_e n_e$$

DETAILED BALANCING

$$\frac{n_{ijk}}{n_{jk}} = \frac{g_{ijk}}{U_{jk}(T)} \exp\left\{-\frac{\chi_{ijk}}{kT}\right\}$$

$$Y_{jk} = \frac{n_{j+1,k}}{n_{jk}} = \frac{C_\Phi}{n_e} \frac{U_{j+1}(T)}{U_{jk}(T)} T^{3/2} \exp\left\{-\frac{\chi_{Ijk}}{kT}\right\}$$

$$f_{jk} = \frac{n_{jk}}{n_k} = \frac{P_{jk}}{S_k} \qquad \text{or} \qquad f_{jk} = f_{j-1,k} Y_{j-1,k} \quad \text{with} \quad f_{1k} = \frac{1}{S_k}$$

$$P_{jk} = P_{j-1,k} Y_{j-1,k} \quad \text{with} \quad P_{1k} = 1 \qquad \text{and} \qquad S_k = \sum_{j=1}^{k+1} P_{jk}$$

CHARGE DENSITY CONSERVATION

$$n_{e,jk} = (j-1) n_{jk} = (j-1) f_{jk} n_k$$

$$n_e = (n - n_e) \sum_k \alpha_k \sum_{j=1}^{k+1} (j-1) f_{jk},$$

$$f_{jk} = \frac{n_{jk}}{n_k}$$

EQUATION OF STATE AND HYDROSTATIC PRESSURE

$$P = P_g + P_r = (P_{\rm N} + P_e) + P_r$$

$$P_{\rm N} = n_{\rm N} kT = \frac{k}{\mu_{\rm N} m_{\rm a}} \rho T\,, \quad P_e = n_e kT = \frac{k}{\mu_e m_{\rm a}} \rho T\,, \quad P_r = \frac{a}{3} T^4$$

$$\mu_{\rm N} = \left[\sum_k \left(\frac{x_k}{A_k} \right) \right]^{-1}\,, \quad \mu_e = \left[\sum_k \left(\frac{x_k}{A_k} \right) \sum_{j=1}^{k+1} (j-1) f_{jk} \right]^{-1}$$

$$\frac{dP}{dx} = -g\rho\,, \quad g = \frac{GM_*}{R_*^2}$$

OPTICAL DEPTH RELATIONS

$$\Delta\tau = \chi'(\rho, T)\rho\Delta x$$

$\chi'(\rho, T) =$ Rossland Mean Mass Absorption Coefficient, from OPAL table

$$\Delta P = g \frac{\Delta\tau}{\chi'(\rho, T)}$$

$$\frac{T(\tau)}{T_{\rm eff}} = \left[\frac{3}{4} \left\{ \tau + q(\tau) \right\} \right]^{1/4} \equiv \frac{1}{p(\tau)},$$

$q(\tau) =$ Hopf Function, from Table (hand out)

FLUX

$$\frac{F_\alpha(\tau)}{F} = \frac{4\pi k^4}{h^3 c^2 \sigma} \alpha^3 \beta(\tau)$$

$$\alpha = \frac{h\nu}{kT_{\rm eff}} = \frac{hc}{\lambda k T_{\rm eff}}$$

$$F = \sigma T_{\rm eff}^4$$

$$\beta(\tau) = \int_\tau^\infty \frac{E_2(t-\tau)dt}{\exp\{\alpha p(\tau)\} - 1} - \int_0^\tau \frac{E_2(\tau - t)dt}{\exp\{\alpha p(\tau)\} - 1}$$

$$E_n(y) = \int_1^\infty \frac{\exp\{-wy\}}{w^n} dw$$

ALGORITHMIC STEPS

(a little help)

1. Your model inputs are
   (i) effective temperature, $T_{\rm eff}$
   (ii) surface gravity $g = GM_*/R_*$
   (iii) composition, mass fractions X, Y, Z
   (iv) Rosseland mean opacity table (for given X, Y, Z)
   (v) boundary condition: total pressure at the top layer, P_1

2. Obtain the boundary condition from the plots from the class notes; be sure to scale by $P_1 \propto g^{2/3}$ as appropriate for your choice of star.

3. Set up the τ_i and $T_i = T(\tau_i)$ arrays (the optical depth and temperature grid); where $\tau_1 = 10^{-3}$. Use a τ grid that samples τ ten times per unit decade.

4. Using P_1, solve the detailed balance of the $i = 1$ layer, obtaining $n_e(\tau_1)$ and $\rho_1 = \rho(\tau_1)$. Compute $\chi'(\rho_1, T_1)$ by interpolating on the the opacity table. You have solved the top layer.

5. (τ_i loop): increment i.

6. Make the initial *estimate* of the pressure, P'_i, in next layer i based upon the opacity of the above layer, $i-1$

$$P'_i = P_{i-1} + g\frac{(\tau_i - \tau_{i-1})}{\chi(\rho_{i-1}, T_{i-1})}.$$

7. (pressure convergence loop): using the guess P'_i, solve the detailed balance of layer i, obtaining your current estimate of $\rho'_i = \rho(\tau_i)$. Compute $\chi'(\rho'_i, T_i)$ by interpolating on the the opacity table.

8. Refine the pressure guess by averageing the pressure gradient. Do this using the average opacity of layer i for your current estimate (i.e., $\chi'(\rho'_i, T_i)$ for P'_i and ρ'_i) and the above (solved) layer $i-1$,

$$P''_i = P_{i-1} + 2g\frac{(\tau_i - \tau_{i-1})}{\chi'(\rho_{i-1}, T_{i-1}) + \chi'(\rho'_i, T_i)}.$$

   This new pressure is an improved estimate of the hydrostatic gradient.

9. Check for convergence: is $|P''_i - P'_i|/P'_i \leq \epsilon$, where ϵ is your specified tolerance level?

   If "YES", then you have achieved hydrostatic equilibrium. Set $P_i = P''_i$ and solve the detailed balance for the layer; store or print all quantities and go to step 5.

   If "NO", then set $P'_i = P''_i$ and go to step 7; repeat until step 9 provides a "YES".