N.M.S.U. Astronomy Department: Cumulative Exam #375 (Solution Set)
1^{st} December 2012 – Nicole Vogt

1. How far would you need to shift the Moon away from the Earth for the center of mass of the Earth-Moon system to lie outside of the Earth (state the current and revised distance between the Earth and the Moon)? (10 pts)

   We begin with the equation for the center of mass for the Earth-Moon system. If r_e is the distance between the center of the Earth and the center of mass, and r_m is the distance between the center of the Moon and the center of mass, then for masses m_e and m_m, and a Earth-Moon separation of d,

$$r_e\, m_e \;=\; r_m\, m_m, \text{ where } r_e \,+\, r_m \;=\; d \;=\; 384{,}000\,\text{km}.$$

   Solving for r_e,

$$r_e \;=\; \left(\tfrac{m_m}{m_e}\right) r_m \;=\; \left(\tfrac{m_m}{m_e}\right) (d \,-\, r_e)$$

   and

$$r_e \;=\; \left(\tfrac{1}{1+m_e/m_m}\right) d \;=\; 4{,}660\,\text{km}$$

   for $m_e = 5.98 \times 10^{27}$ grams and $m_m = 7.35 \times 10^{25}$ grams, which is clearly less than $R_\oplus$ = 6,370 km. Raising r_e up to $R_\oplus$,

$$d \;=\; \left(1 \,+\, \tfrac{m_e}{m_m}\right) R_\oplus \;=\; 525{,}000\,\text{km}.$$

2. If an asteroid at $r = 4$ AU occults a solar-type star at 10 parsecs when at opposition, how large is the asteroid? (15 pts)

   The asteroid must completely block the star, so for an asteroid diameter of d,

$$\tfrac{d}{3/206265\,\text{pc}} \;=\; \tfrac{2R_\odot}{10\,\text{pc}}$$

   where there are 206265 AU per parsec and the asteroid lies 3 AU from Earth, and so

$$d \;=\; \left(\tfrac{2\times 3}{10\times 206265}\right) R_\odot \;=\; 2.90\times 10^{-6}\, R_\odot \;=\; 202{,}000\,\text{cm} \;=\; 2.02\,\text{km}$$

   for $R_\odot = 6.96 \times 10^{10}$ cm. The diameter of the asteroid must thus be greater than or equal to 2 kilometers.

3. What if our solar system were actually a binary stellar system? Mass extinction events in the fossil record suggest that if the dreaded Nemesis exists, it has a period of 30 Myr. If Nemesis has a mass of $0.1\,M_\odot$, estimate its orbital semi-major axis, absolute and apparent magnitudes, and rate of angular shift in position as viewed from Earth. Where does this place it in the solar system? (20 pts)

   We can solve for the semi-major axis A of the orbit as follows. For a body with orbital period P,

$$P^2 = \left[\frac{(2\pi)^2}{G\,(M_\odot + 0.1M_\odot)}\right] A^3.$$

and

$$A = \left[\left(\frac{P}{2\pi}\right)^2 (1.1M_\odot)\,G\right]^{1/3} = 1.49 \times 10^{18}\,\text{cm} = 99,000\,\text{AU}.$$

for G = 6.67×10^{-8} cm^3 per gram per sec^2 and $M_\odot = 1.99 \times 10^{33}$ grams, placing Nemesis in the Oort Cloud.

We estimate the luminosity from the mass, knowing that

$$L\,(L_\odot) = [M\,(M_\odot)]^4 = 0.1^4\,L_\odot = 10^{-4}\,L_\odot$$

and then derive the absolute magnitude.

$$\frac{L}{L_\odot} = 10^{-(M-M_\odot)/2.5}, \text{ and so}$$

$$M = M_\odot - 2.5\log(L/L_\odot) = 4.8 - 2.5(-4) = 14.8.$$

Using the distance modulus,

$$m - M = 5\log(d) - 5$$

and so

$$m = M + 5\log(d) - 5 = 14.8 + 5\log(99,000/206265) - 5 = 8.2.$$

Nemesis will cover 360° every 30 Myr, and so the angular rate of change α is

$$\alpha = \frac{360^\circ}{3\times10^7\,\text{yr}} = 1.2\times10^{-5}\,\text{degrees per year, or } 0.04\,\text{arcseconds per year.}$$

4. Assume that the space surrounding the Sun is filled with solar-type stars, with a density of one star per cubic parsec. How many stars should be visible to the naked eye in a dark sky? Compare your answer to typical observations, and critique the initial assumptions. (15 pts)

   We can computer the number of solar-type stars visible down to sixth magnitude as follows. Solving the distance modulus for d,

   $$d = 10^{(m-M+5)/5} = 10^{(6.0-4.8+5)/5} = 17.4\,\text{parsec}.$$

   Within a sphere of radius $R = 17.4$ parsecs, the number of stars N is then

   $$N = \tfrac{4}{3}\pi\rho R^3 = \tfrac{4}{3}\pi(1)17.4^3 = 22,000\,\text{stars}.$$

   We expect roughly 6,000 stars to be visible in a dark sky. Our initial estimate of a stellar density of 1 star per cubic parsec is quite high (by roughly a factor of seven), and is the primary cause of the over-estimate.

5. What fractional decrease in flux would occur for a transit of the Earth in front of the Sun, viewed from a great distance away along the plane of the ecliptic? How long would the transit last? (15 pts)

The decrease in flux will scale with the decrease in presented surface area, so

$$\frac{f}{f_0} = \frac{\pi}{\pi}\left(\frac{R_\oplus}{R_\odot}\right)^2 = \left(\frac{6.37\times10^8\,\mathrm{cm}}{6.96\times10^{10}\,\mathrm{cm}}\right)^2 = 8.38\times10^{-5} = 0.00838\%.$$

The transit time is a function of the solar radius $R_\odot$ and the Earth's speed V_e, so

$$T = \frac{2R_\odot}{V_e} = \frac{2R_\odot}{2\pi R_{orb}/T_{orb}} = \left(\frac{R_\odot}{R_{orb}}\right)\frac{T_{orb}}{\pi}$$

$$T = \left(\frac{6.96\times10^{10}\,\mathrm{cm}}{1.50\times10^{13}\,\mathrm{cm}}\right)\frac{\pi\times10^7\,\mathrm{sec}}{\pi} = 46,400\,\mathrm{seconds} = 12.9\,\mathrm{hours}.$$

6. (a) Consider a body of mass m and radius r orbiting another body with mass M at a distance of R. If m is self-bound by its own gravity, derive a relationship so that the tidal force just balances the binding force. (15 pts)

   (b) How close could the Sun orbit to a massive black hole (M $= 10^6\,M_\odot$) before being tidally disrupted? (5 pts)

   (c) How large is the black hole (what is its Schwarzchild radius)? (5 pts)

   (a) The self-gravitational force F_b is

$$F_b = G\left(\frac{m}{r^2}\right)$$

while the tidal force per unit mass m is (using a Taylor Series expansion)

$$F_t = GM\left(\frac{1}{R^2}\right) - GM\left[\frac{1}{(R+r)^2}\right] = GM\left(\frac{1}{R^2}\right)\left[1-\left(1-\frac{2r}{R}\right)\right] = GM\left(\frac{2r}{R^3}\right).$$

Equating the two forces, we see that

$$\frac{m}{M} = 2\left(\frac{r}{R}\right)^3.$$

   (b) For a super-massive black hole and a too-curious Sun,

$$R\,(R_\odot) = \left(\frac{M}{2m}\right)^{1/3} = (2\times10^6)^{1/3} = 126\,R_\odot = 8.77\times10^{12}\,\mathrm{cm}.$$

   (c) The black hole has a Schwarzchild radius of

$$R_{Sch} = \frac{2GM}{c^2} = \left[\frac{2\cdot6.67\times10^{-8}\cdot10^6\cdot1.99\times10^{33}}{(3\times10^{10})^2}\right]\mathrm{cm} = 2.95\times10^{11}\,\mathrm{cm} = 2.95\times10^6\,\mathrm{km}.$$