

## Cume 428

More than 60% garanties the pass.

**(1) Mass distribution of our Galaxy.** Assume a flat rotation curve for our galaxy with the circular velocity of  $v_{\text{circ}} = 250 \text{ km/s}$ . Distance of the Sun to the center is 10 kpc. The virial radius is 250 kpc.

**(1a) 10pt** What is the mass of the galaxy inside the virial radius? Using the ptovided information above, derive the mass. Give the answer in units of the solar mass.

*Answer:*

*Object moving on a circle at a distance  $r$  has velocity  $v_{\text{circ}}^2/r = g$ , where  $g(r)$  is the gravitational acceleration. With good accuracy we can write  $g = GM(r)/r^2$ , where  $M(r)$  is the mass inside a sphere of radius  $r$ . Thus, the circular velocity  $v_{\text{circ}}$  is related to the mass as  $v_{\text{circ}} = \sqrt{GM(r)/r}$ . Constant circular velocity implies  $M(r) \propto r$ . Slightly re-writing this relation and estimating it at virial radius, we find*

$$M(r) = v_{\text{circ}}^2 r_{\text{vir}} / G = 3.6 \times 10^{12} M_{\odot}. \quad (1)$$

*This estimate is a factor of 2-3 to big because we assumed that the circular velocity stays flat all the way to the virial radius. In reality it start to decline outside the solar radius resulting in smaler mass estimate.*

**(1b) 10pt** What is the escape velocity at the solar distance? Assume that there is no mass beyond the virial radius. Give the answer in units of km/s.

*Answer:*

*The escape velocity is defined by condition that the total specific energy of a particle is equal to zero:  $v^2/2 + \phi = 0$ , where  $\phi$  is the gravitational potential at the solar radius. We will estimate the gravitation potential assuming that it is spherically symmetric, which is actually quite a good approximation. The potential has two terms: the potential due to the mass  $M_0$  inside the solar radius  $R_0$  and potential due to the layers of mass at larger radii. The gravitational potential is the work done by the force of gravity when object of unit mass is displaced from radius  $r$  (solar radius in our case) to infinity:*

$$\phi(r) = - \int_r^{\infty} \frac{GM(r)}{r^2} dr = - \frac{GM_0}{R_0} \left[ \int_{R_0}^{R_{\text{vir}}} \frac{dr}{r} + R_{\text{vir}} \int_{R_{\text{vir}}}^{\infty} \frac{dr}{r^2} \right] \quad (2)$$

*Here we used  $M(r) = M_0(r/R_0)$  for  $r < R_0$  and  $M(r) = M_0(R_{\text{vir}}/R_0)$  for larger radii. Taking the integrals and noticing that  $GM_0/R_0 = v_{\text{circ}}^2$ , we finally write:*

$$v_{\text{esc}} = \sqrt{2v_{\text{circ}}^2 [1 + \ln(R_{\text{vir}}/R_0)]} = v_{\text{circ}} \sqrt{2(1 + \ln(25))} = 725 \text{ km/s}. \quad (3)$$

**(2) 20pt Number of satellites around the Milky Way.** How many satellites our Galaxy should have within the virial radius? Estimate the number using the following information. Assume that the average number density of dwarf galaxies in the Universe is 1 per cubic megaparsec. The correlation function of dwarf galaxies is described as a power-law:  $\xi(r) = (r/6\text{Mpc})^{-1.8}$ . See parameters of MW above. Remember that the correlation function is the probability in excess of random to find a galaxy at a distance  $r$  from another galaxy.

*Answer:*

*The number of galaxies  $dN$  expected in a volume  $dV$  at a distance  $r$  from the center of the Milky Way is*

$$dN = n(1 + \xi(r))dV, \quad (4)$$

*where  $n$  is the number-density of dwarf galaxies. Note that the first term in the parenthesis gives the expected number for random objects with average density  $n$ . The second term tells how many objects are expected in addition to the random number. This is the definition of the correlation function. We can integrate this expression from  $r = 0$  to  $r = r_{\text{vir}}$  to find how many dwarfs (called satellites) are expected inside the virial radius:*

$$N = 4\pi n \int_0^{r_{\text{vir}}} (1 + \xi(r))r^2 dr = 4\pi n \left[ \frac{1}{3}r_{\text{vir}}^3 + \frac{6^{1.8}}{1.2}r_{\text{vir}}^{1.2} \right] \quad (5)$$

*This gives about 50 satellites, which is close to observed satellites of the Milky Way with circular velocities above  $\sim 10$  km/s.*

**(3) Virial theorem** simply states that  $2K + W = 0$ , where  $K$  and  $W$  are the kinetic and potential energies.

**(3a) 5pt** Define all the quantities. Kinetic and potential energies of *what*? Are only observed quantities involved? Say, we can measure only bright stars in globular clusters. How do we account for all stars? How do we treat the dark matter?

*Answer:*

*The kinetic and potential energies are total values for all mass in the system – visible or not.*

$$K = \frac{1}{2} \sum_i m_i V_i^2, \quad W = -\frac{1}{2} \sum_{i,j} \frac{Gm_i m_j}{r_{ij}} \quad (6)$$

**(3b) 10pt** How do we use the virial theorem to measure the “virial mass”?

*Answer: Rewrite the energies:*

$$K = M_{\text{tot}} \frac{\langle V^2 \rangle}{2} \quad W = -GM_{\text{tot}}^2 \left\langle \frac{1}{R} \right\rangle. \quad (7)$$

The virial theorem now states that:

$$GM_{\text{tot}} \langle \frac{1}{R} \rangle = \langle V^2 \rangle. \quad (8)$$

One needs to measure the rms velocity and the average  $1/R$  of a representative population of objects.

**(3c) 10pt** Write expression for a virial mass in the form  $M_{\text{vir}} = (\text{combinations of observable quantities})$ . How do you modify this expression to include the dark matter?

*Answer:*

We can only measure the line-of-sight (los) velocities. Assuming that velocities are isotropic statistically  $\langle V_{3d}^2 \rangle = 3\langle V_{\text{los}}^2 \rangle$ . The average  $1/R$  can be estimated by averaging pair-wise  $1/R$  projected distances of observed objects. This gives:

$$M_{\text{tot}} = 3\langle V_{\text{los}}^2 \rangle \langle R \rangle / G. \quad (9)$$

This gives the total mass of the system that includes the dark matter in spite of the fact that we measure only visible population. However, this population should be unbiased in the the sense that it gives the true measures of rms velocities and spatial  $\langle 1/R \rangle$ .

**(3d) 5pt** There are two identical globular clusters. One is at rest and another moves with velocity of 100 km/s. If we apply the virial theorem, which one will give larger virial mass? Explain your reasoning. Answers “the first” or “the second” do not count unless your explanation is correct.

*Answer:*

The total (drift) velocity should not be included in the estimate of the kinetic energy. So, the virial masses for moving and stationary clusters are the same. The reason is in the fact that the virial theorem is only applicable to stationary systems: those that have the second derivative over time of the moment of inertia equal to zero. So, formally the virial theorem is not applicable to the moving cluster. However, by changing the reference frame to the one that moves with cluster, we can have the system stationary again.

#### **(4) Hydrostatic equilibrium.**

**(4a) 20pt** How does the density and temperature change with the height in the Earth’s atmosphere at elevations up to 10 km? Assume that the atmosphere is isentropic (has constant entropy). Temperature and density at the surface are  $T_0$  and  $\rho_0$ . The answer should be in the form of equations  $T = \dots$ ,  $\rho = \dots$  derived using the hydrostatic equation and gas properties.

*Answer:*

The equation of hydrostatic equilibrium is written as

$$g(r) = -\frac{1}{\rho} \frac{dP}{dr}, \quad (10)$$

where  $g$  is the gravitational acceleration,  $\rho$  and  $P$  are gas density and pressure. For adiabatic gas  $P = A\rho^\gamma$ . Find  $A$  from on-ground values:

$$P_0 = A\rho_0^\gamma, P_0 = \rho_0 \frac{kT_0}{\mu m_H} \rightarrow A = \frac{kT_0}{\mu m_H} \rho_0^{1-\gamma}. \quad (11)$$

The left-hand-side term of the hydrostatic equation takes form:

$$\frac{1}{\rho} \frac{dP}{dr} = A\gamma\rho^{\gamma-2} \frac{d\rho}{dz} = A \frac{\gamma}{\gamma-1} \frac{d\rho^{\gamma-1}}{dz}. \quad (12)$$

Now we find solution of the hydrostatic equation:

$$\rho^{\gamma-1} = \rho_0^{\gamma-1} - \frac{gz}{A} \frac{\gamma}{(\gamma-1)}. \quad (13)$$

Rewrite it as expression for the density:

$$\rho(z) = \rho_0 \left[ 1 - \frac{z}{z_H} \right]^{\frac{1}{(\gamma-1)}}, \quad (14)$$

where the scale height of the atmosphere is

$$z_H = \frac{\gamma}{\gamma-1} \frac{kT_0}{g\mu m_H}. \quad (15)$$

Find temperature using the equation of state  $P = A\rho^\gamma = \rho \frac{kT}{\mu m_H}$ :

$$T(z) = T_0 \left[ \frac{\rho}{\rho_0} \right]^{(\gamma-1)} = T_0 \left[ 1 - \frac{z}{z_H} \right], \quad (16)$$

**(4b) 10pt** We can estimate the mass distribution (mostly dark matter)  $M(r)$  in a cluster of galaxies by measuring x-ray flux and temperature of gas. The gas temperature is nearly constant  $T_{\text{gas}} = \text{const}$ . Assume that x-ray measurements give us gas density profile  $\rho_{\text{gas}}(r)$ . Further assume that the gas is in hydrostatic equilibrium in a spherically symmetric cluster. Write an expression for the total mass distribution  $M(r)$ .

*Answer:*

In this case the gravitational acceleration is  $g = GM(r)/r^2$ . For isothermal ideal gas the derivative of the pressure with radius can be written as:

$$\frac{dP}{dr} = \frac{kT}{\mu m_H} \frac{d\rho_{\text{gas}}}{dr}, \quad (17)$$

where  $\mu$  and  $m_H$  are the molecular weight and hydrogen mass. Putting these relations to the equation of the hydrostatic equilibrium we find:

$$M(r) = - \frac{kT}{\mu m_H} \frac{r}{G} \frac{d \ln \rho_{\text{gas}}}{d \ln r}, \quad (18)$$

Equation of hydrostatic equilibrium:

$$g(r) = -\frac{1}{\rho} \frac{dP}{dr}, \quad (19)$$

Ideal gas equation:

$$P = nkT = \frac{\rho kT}{\mu m_H}. \quad (20)$$

Adiabatic gas equation:

$$P = A\rho^\gamma, \quad \gamma = c_p/c_v. \quad (21)$$

$$M_\odot = 2 \times 10^{33} \text{g}.$$

$$1 \text{kpc} = 3.085 \times 10^{21} \text{cm}.$$

$$\text{Grav.constant } G = 6.672 \times 10^{-8} \text{cm}^3 \text{g}^{-1} \text{sec}^{-2}.$$

