Cume #461
The Constant-Density Universe
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April 23, 2022

This exam covers idealized problems that are tractable because we will assume a **constant density**. This will include either a constant mass density in space and time: $\rho(\boldsymbol{r}, t) = \rho_0$, or a constant number density: $n(\boldsymbol{r}, t) = n_0$. The topics span from polytropes to cosmology! Each of the four problems has a short calculation and often some required description of the significance or interpretation of the problem you are solving. The *gauranteed passing grade* is 75%.

Please use a new page for each numbered problem. Show all work clearly and please write legibly, and if you can't solve something completely, at least give an idea of how you might go about it. Make sure you are careful to answer ALL parts of each question. Don't spend too much time in the beginning on one question, move on and try them all and then come back if you need to.

This is a remote exam and you are not allowed to use any materials, such as books or websites or your computer (except for emailing a question if needed). You may use a calculator for any arithmetic computations only, not for plotting or algebra or for storing equations.

If any clarification questions are needed, just email, call (505.431.3557), or Canvas chat. Good luck!

Some numbers you may need:

$$
\begin{aligned}
R_\odot &= 6.96 \times 10^{10}\,\mathrm{cm} \\
m_{\mathrm{proton}} &= 1.67 \times 10^{-27}\,\mathrm{kg}; \\
G &= 6.674 \times 10^{-20}\,\mathrm{km}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}; \\
1\,\mathrm{parsec} &= 3.086 \times 10^{16}\,\mathrm{m};
\end{aligned}
$$

Here are a few other useful things you may need:

- Definition of mass density:
$$\mathrm{d}m = \rho\,\mathrm{d}V = 4\pi\rho r^2 \mathrm{d}r,$$
where the second equality holds in the spherically-symmetric case.
- The gravitational potential energy in a small shell of a spherically-symmetric object can be written
$$\mathrm{d}\Phi(r) = -\frac{Gm(r)}{r}\,\mathrm{d}m,$$
where $m(r)$ is the interior mass to point r and $\mathrm{d}m$ is the element of mass at point r.
- The solid angle subtended by a star of radius R_* and a distance r is
$$\mathrm{d}\Omega = \frac{\pi R_*^2}{r^2},$$
in units of steradians (radians^2).
- A simplified Friedmann equation (describing how the scale factor of the universe evolves in time) is
$$H^2 = \frac{8\pi G}{3}\rho + \frac{k}{a^2},$$
where H is the Hubble constant, ρ is the mass density of the universe, k is the curvature, and $a = a(t)$ is the scale factor of the expanding universe. For simplicity, k can only take values of $k = \pm 1$ or $k = 0$ for curved or flat. To be clear, the scale factor is related to the Hubble constant by $\dot{a}/a = H$.

1. (5 points). Polytropes. Self-gravitating spheres can be modeled by polytropes. A very useful expression is the one for the gravitational potential energy of a polytropic star of arbitrary index n, mass M, and radius R:

$$\Phi(n) = -\frac{3}{5-n}\frac{GM^2}{R}. \tag{1}$$

Recall that polytropes obey a general equation of state of the form

$$p = K\rho^{\gamma} = K\rho^{1+1/n}, \tag{2}$$

where p is pressure. A polytropic model for $n = 0$ has **constant mass density** since $\rho = \rho_0\theta^n$, where θ is the interior spatial coordinate and ρ_0 is some reference density. Show that the expression for the gravitational potential energy predicted by Eq. (1), $\Phi(n = 0)$, is exactly the one you get by computing directly the gravitational potential energy for a constant density model.

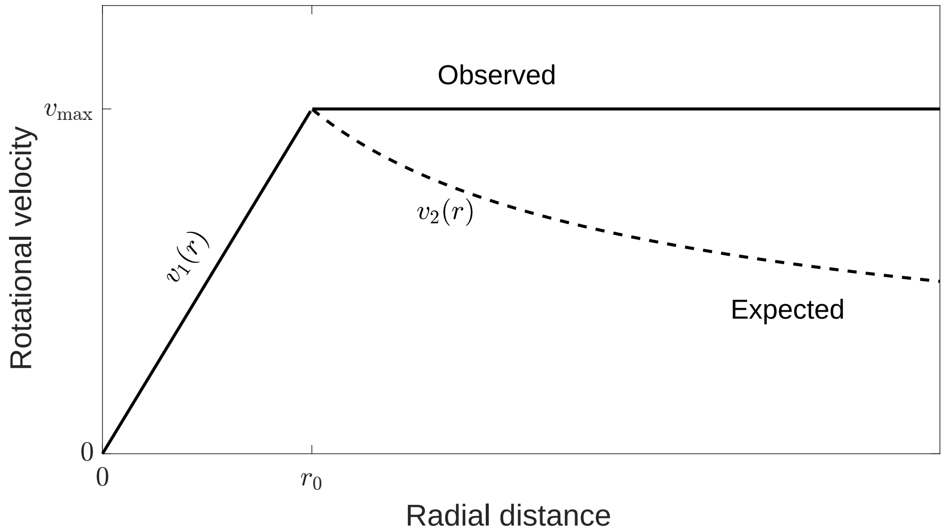

Figure 1: See Problem 2. The solid lines illustrate an idealized rotation curve of a spiral galaxy that one might observe. Both theory and observations agree on the form of $v_1(r \leqslant r_0)$. However, beyond r_0, they disagree, with the dashed line $v_2(r)$ being the curve one might expect from simple theory.

2. (12 points). Galaxies. Observed galaxy rotation curves might look as in Figure 1. Basic theory in the early 20th century theory predicted a different curve at large distances, which we can derive. Assume a spiral galaxy, which has **constant mass density** as a function of distance from its center until a point at which it goes to zero. Therefore, $\rho(r) = \rho_0 = \text{const}$ for $r \leqslant r_0$ and $\rho(r) = 0$ for $r \geqslant r_0$.

   (a) (3 points) Imagine you have access to any observational tools you choose. How would you go about measuring the rotation curve of a distant spiral galaxy?

   (b) (3 points) Explain historically why the difference between the observed curve and what was expected at large distances is so significant? What were the implications? Can you name anyone involved in this research at the time?

   (c) (6 points) Derive explicitly each of the functional forms for $v_1(r)$ and $v_2(r)$ under the **constant-density** assumption given above.

3. (12 points). Cosmology. Let's say the night sky were uniformly bright from starlight instead of how we see it. From how far away would the light have to be arriving for this to be the case? Let's call this distance r_0.

   (a) (6 points) Assume that there is a **constant number density** of stars in the universe n_*, and that all stars have the same radius R_*. Find an expression for r_0 in terms of only these two parameters (and any constants). *Hint: first consider a thin spherical shell (thickness* $\mathrm{d}r$*) a distance r away from Earth and compute an expression for the fraction of the sky the stars in that shell will cover, and then integrate that shell out to some distance r_0. Also, note that there is a completely analogous number density equation to the mass density equation given on the first page.*

   (b) (3 points) Pick a "reasonable" galactic value (i.e., in our galaxy) for n_* and R_*. Using your expression from part (a), what do you calculate for r_0? Even if you did not get a good expression, pick these two values.

   (c) (3 points) A more rigorous calculation gives a value of about $r_0 \approx 1 \times 10^{16}$ pc. Using this value, why is the night sky not uniformly bright? Explain.

4. (16 points). Cosmology. A popular early cosmological model posited an unchanging universe, such that the **mass density of the universe is constant** at all times. In an expanding universe, which was accepted already at that time, this model therefore requires the continuous creation of matter. Let's call this *Model A*.

   (a) (2 points) What is the name of *Model A*, or, if you don't recall, what are some of the properties of the model, as compared to the current Big-Bang model, which we can call *Model B*? Do you recall any of the proponents of *Model A*?

   (b) (2 points) Describe one cosmological observation that refutes *Model A* or any of its claims?

   (c) (4 points) Similar to *Model A*, *Model B* already claims that, under certain conditions, our universe has a homogeneous and isotropic mass density in **space** (but not time). What is the main condition for this bold claim, what does homogeneity and isotropy in space mean, and what is this concept known as?

   (d) (8 points) Consider a simple, flat (no curvature) universe without a cosmological constant and $\Omega_{\text{mass}} = \rho/\rho_{\text{crit}} = 1$ at all times. Assume the Hubble constant is indeed constant at today's measured value. Show that, for this early cosmological model to work and maintain a constant mass density, the amount of matter required to be created is about 1 hydrogen atom per km^{-3} per year. You will need to supply a decent value of the Hubble constant here.