Cume #440 - Solutions
Math in Astronomy
Jason Jackiewicz
February 15, 2020

This exam deals with some fundamental, basic mathematics that can occur frequently in astrophysics research. In fact, all of these problems are loosely related to topics that have arisen in my daily research or course preparation over the past 18 months and scribbled in my notebooks, in one form or another. These topics cover things that aren't always trivial to just "look up" to find the right answer. The *anticipated passing grade* is 75%.

Show all work clearly and please write legibly, and if you can't solve something completely, at least give an idea of how you might go about it. Make sure you are careful to answer ALL parts of each question. Don't spend too much time in the beginning on one question, move on and try them all and then come back if you need to. No calculators allowed please. Good luck!

Here are a few things you may need.

- Matrix multiplication.
$$\boldsymbol{C} = \boldsymbol{A}\boldsymbol{B}; \quad c_{ij} = \sum_k a_{ik} b_{kj}. \tag{1}$$
- Taylor expansion.
$$f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + \dots \tag{2}$$
- Internal energy density of an ideal gas (energy per unit volume):
$$u = \frac{3}{2} n k_{\mathrm{B}} T, \tag{3}$$
where the terms are particle density, Boltzmann's constant, and temperature.
- Specific internal energy (energy per unit mass): $U = u/\rho$, where ρ is mass density.
- $\mathrm{d}m = 4\pi\rho r^2 \mathrm{d}r$
- $P = n k_{\mathrm{B}} T$.
- The continuity equation, which expresses mass conservation of a fluid in some given volume:
$$\boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0, \tag{4}$$
where ρ is mass density and $\boldsymbol{v}$ is the fluid velocity. $\rho\boldsymbol{v}$ is the mass flux.
- Divergence operator in spherical coordinates (r, θ, ϕ) for generic vector $\boldsymbol{A} = (A_1, A_2, A_3)$:
$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 A_1\right) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta}(\sin\theta A_2) + \frac{1}{r\sin\theta} \frac{\partial A_3}{\partial \phi} \tag{5}$$
- Curl operator in spherical coordinates (r, θ, ϕ) for generic vector $\boldsymbol{A} = (A_1, A_2, A_3)$:
$$\begin{aligned} \boldsymbol{\nabla} \times \boldsymbol{A} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_3) - \frac{\partial A_2}{\partial\phi}\right]\hat{\boldsymbol{e}}_1 \\ &+ \left[\frac{1}{r\sin\theta}\frac{\partial A_1}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_3)\right]\hat{\boldsymbol{e}}_2 + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_2) - \frac{\partial A_1}{\partial\theta}\right]\hat{\boldsymbol{e}}_3 \end{aligned} \tag{6}$$
- Discrete Fourier transform of a signal $f(t)$ and its inverse, where in this case ω is the angular frequency:
$$f(\omega) = \int_{t_1}^{t_2} f(t) e^{\mathrm{i}\omega t}\, \mathrm{d}t, \tag{7}$$
$$f(t) = \int_{\omega_1}^{\omega_2} f(\omega) e^{-\mathrm{i}\omega t}\, \mathrm{d}\omega \tag{8}$$

1. (10 points).

(a) Let $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 2 & 1 \end{bmatrix}$. Compute $\boldsymbol{C} = \boldsymbol{AB}$

Answer: This is a 2x2 matrix. $\boldsymbol{C} = \begin{bmatrix} -1 & -1 \\ -2 & -2 \end{bmatrix}$.

Or,

$$\begin{aligned} c_{11} &= a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} = -1 \\ c_{12} &= a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} = -1 \\ c_{21} &= a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} = -2 \\ c_{22} &= a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} = -2 \end{aligned}$$

(b) Let $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$. Compute $\boldsymbol{C} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{B}$, where T is the transpose.

Answer: This should be rewritten as $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$.

It is a 3x3 matrix. $\boldsymbol{C} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix}$. Or,

$$\begin{aligned} c_{11} &= a_{11}b_{11} = 1 \\ c_{12} &= a_{11}b_{12} = 0 \\ c_{13} &= a_{11}b_{13} = -1 \\ c_{21} &= a_{21}b_{11} = 2 \\ c_{22} &= a_{21}b_{12} = 0 \\ c_{23} &= a_{21}b_{13} = -2 \\ c_{31} &= a_{31}b_{11} = 1 \\ c_{32} &= a_{31}b_{12} = 0 \\ c_{33} &= a_{31}b_{13} = -1 \end{aligned} \tag{9}$$

2. (10 points).

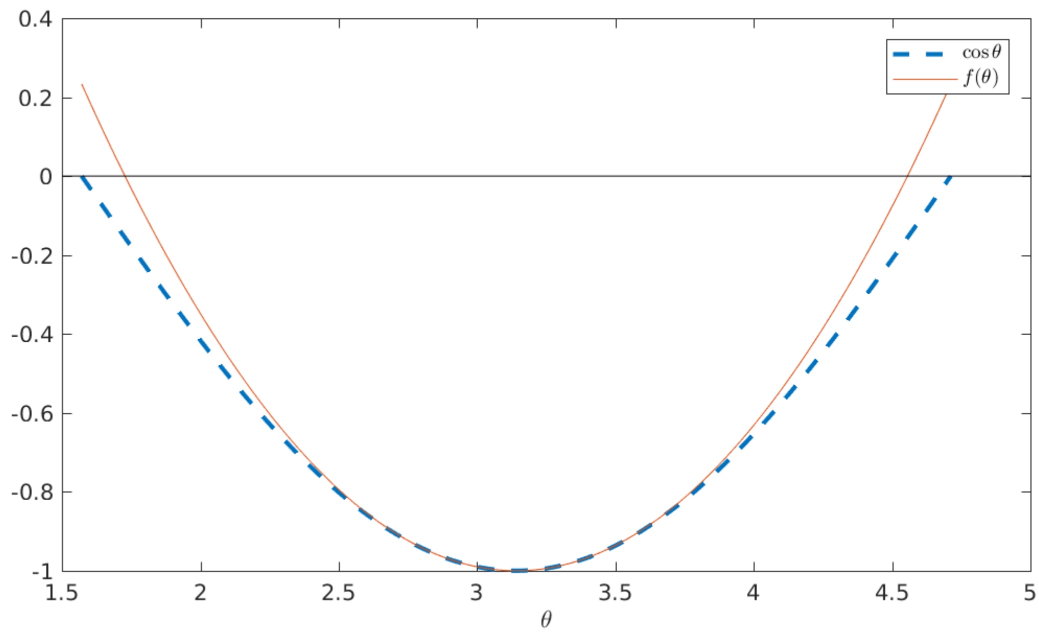
(a) Find a second-order approximation to the function $f(\theta) = \cos\theta$ about the point $\theta = \pi$.

Answer: From Taylor's theorem

$$\begin{aligned} f(\theta) &\approx \cos\pi - \sin\pi(\theta-\pi) - \frac{1}{2}\cos\pi(\theta-\pi)^2 && (10)\\ &\approx -1 - 0 + \frac{1}{2}(\theta-\pi)^2 && (11)\\ &\approx -1 + \frac{1}{2}(\theta-\pi)^2. && (12)\end{aligned}$$

(b) Now sketch $\cos\theta$ and your approximation from $\theta = \pi/2$ to $\theta = 3\pi/2$ as accurately as possible (primarily the mid and end points).

Answer: The approximation works well at π but not at the end points. Where $\cos\theta = 0$ at the end points, our approximation is instead $f(\pi/2, 3\pi/2) = -1 + \pi^2/8 > 0$.



3. (8 points). Compute the value of the definite integral

$$I = \int_{\epsilon}^{1} 9\,x^2 \ln x \,\mathrm{d}x, \tag{13}$$

where ϵ is an extremely small number. Hint: integrate by parts.

<u>Answer</u>:

$$\begin{aligned} u &= \ln x \quad ; \qquad \mathrm{d}v = 9x^2\mathrm{d}x \\ \mathrm{d}u &= \frac{1}{x}\mathrm{d}x \quad ; \qquad v = 3x^3 \end{aligned}$$

$$\begin{aligned} I &= 3x^2 \ln x|_{\epsilon}^{1} - \int_{\epsilon}^{1} 3x^2\,\mathrm{d}x \\ &= 0 - x^3|_{\epsilon}^{1} \\ &= -1. \end{aligned}$$

4. (10 points). The Virial Theorem tells us that the gravitational potential energy (say, $E_{\rm g}$) plus twice the total internal energy (say, $U_{\rm T}$) is zero for a star that's an ideal gas. The Virial Theorem can be derived conveniently from the equation of hydrostatic equilibrium. Do this by multiplying both sides of the equation of hydrostatic equilibrium by $4\pi r^3$ and integrating both sides from 0 to R, i.e., from the star center to the surface. You should be able to arrive at one of the main terms on the right-hand-side of the equation with minimal work. For the left-hand-side, integration by parts may be helpful at one stage. If you use any simplifications or approximations, please point them out.

Answer: The Virial Theorem thus says that $E_{\rm g} + 2U_{\rm T} = 0$. Following the prescription,

$$\begin{aligned}
\frac{\mathrm{d}P}{\mathrm{d}r} &= -\rho g = -\rho\frac{Gm}{r^2} \\
\int_0^R 4\pi r^3 \frac{\mathrm{d}P}{\mathrm{d}r}\,\mathrm{d}r &= -\int_0^R 4\pi r^3 \rho \frac{Gm}{r^2}\,\mathrm{d}r \\
u = 4\pi r^3 \quad &; \quad \mathrm{d}v = \frac{\mathrm{d}P}{\mathrm{d}r}\mathrm{d}r \\
\mathrm{d}u = 12\pi r^2\,\mathrm{d}r \quad &; \quad v = P \\
4\pi r^3 P|_0^R - \int_0^R 12\pi P r^2\,\mathrm{d}r &= -\int_0^R \frac{Gm}{r}4\pi\rho r^2\,\mathrm{d}r \\
P(R) - 0 - \int_0^R 3\frac{P}{\rho}4\pi\rho r^2\,\mathrm{d}r &= -\int_0^M \frac{Gm}{r}\,\mathrm{d}m \\
-3\int_0^M \frac{2}{3}\frac{u}{\rho}\,\mathrm{d}m &= E_{\rm g};\ \ P(R)\to 0 \\
-2\int_0^M U\,\mathrm{d}m &= E_{\rm g} \\
-2U_{\rm T} &= E_{\rm g} \\
E_{\rm g} + 2U_{\rm T} &= 0.
\end{aligned}$$

If, in the above, one did not switch to the mass coordinate on either side, and used the internal energy density instead,

$$\begin{aligned}
-\int_0^R 12\pi P r^2\,\mathrm{d}r &= E_{\rm g} \\
-\int_0^R 12\frac{2}{3}\pi r^2 u\,\mathrm{d}r &= E_{\rm g} \\
-2\int_0^R u 4\pi r^2\,\mathrm{d}r &= E_{\rm g} \\
-2\iiint u\,\mathrm{d}V &= E_{\rm g} \\
E_{\rm g} + 2U_{\rm T} &= 0.
\end{aligned}$$

5. (10 points). Consider a three-dimensional fluid in spherical coordinates that obeys the continuity equation everywhere. The mass flux can then be conveniently expressed as

$$\rho \boldsymbol{v} = \boldsymbol{\nabla} \times \psi \hat{\boldsymbol{\phi}}, \tag{14}$$

where ψ is simply a scalar function that only depends on position $\psi = \psi(r, \theta, \phi)$. ψ is sometimes known as a "stream function."

(a) First, quickly show/describe/argue that if the mass flux satisfies the continuity equation, then it is always possible to write an expression like the right-hand-side of Eq. (14). You don't necessarily have to compute the divergence here.

Answer: If $\boldsymbol{\nabla} \cdot \rho \boldsymbol{v} = 0$, then we can always take the divergence of the curl operator since the curl operation returns a vector perpendicular to $\boldsymbol{\nabla}$, whose divergence is automatically zero.

(b) Now, using Eq. (14), find expressions for each component of the full 3-D velocity field $\boldsymbol{v} = (v_r, v_\theta, v_\phi)$ in terms of the stream function ψ.

Answer:

$$\begin{aligned}
\rho v_r &= \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (\psi \sin\theta) \\
&= \frac{1}{r \sin\theta} \left(\psi \cos\theta + \sin\theta \frac{\mathrm{d}\psi}{\mathrm{d}\theta} \right) \\
&= \frac{1}{r} \frac{\cos\theta}{\sin\theta} \psi + \frac{1}{r} \frac{\mathrm{d}\psi}{\mathrm{d}\theta}.
\end{aligned}$$

$$\begin{aligned}
\rho v_\theta &= -\frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r} (r\psi) \\
&= -\frac{1}{r} \left(\psi + r \frac{\mathrm{d}\psi}{\mathrm{d}r} \right) \\
&= -\frac{\psi}{r} - \frac{\mathrm{d}\psi}{\mathrm{d}r}.
\end{aligned}$$

Then,

$$\rho v_\phi = 0,$$

since there is no component parallel to the $\hat{\phi}$ direction due to the definition of the stream function.

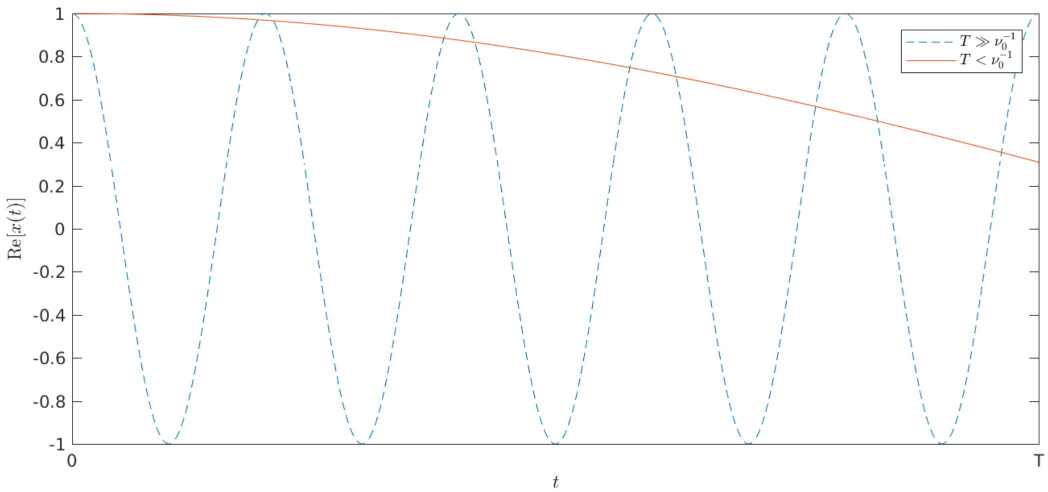
6. (15 points). Say you have recorded a signal from observations from $t = 0$ to $t = T$. The signal has a functional form given by $x(t) = A \exp(-\mathrm{i}\omega_0 t)$, where ω_0 is a single, angular frequency and A is the amplitude.

(a) Assuming that $T \gg 1/\nu_0 \gg (\omega_0/2\pi)^{-1}$, sketch the real part of your signal as accurately as possible, with axes labeled (ν_0 is the corresponding cyclic frequency).

Answer: This is just a cosine function that has ν_0 of oscillations in time T, with amplitude $\pm A$.

(b) Now sketch the real part of the signal assuming that $T < 1/\nu_0 < (\omega_0/2\pi)^{-1}$, again, with everything labeled.

Answer: This is a cosine function that doesn't go through a full period since $P = 1/\nu_0 > T$..



(c) Compute the power spectrum of your signal. The power spectrum is given by $P(\omega) = |x(\omega)|^2$, where $x(\omega)$ is the (complex) Fourier transform of the signal. Show that it's

$$P(\omega) = \frac{2A^2}{(\omega - \omega_0)^2} \left[1 - \cos\left((\omega - \omega_0)T\right)\right]. \tag{15}$$

or, equivalently,

$$P(\omega) = A^2 T^2 \operatorname{sinc}^2 \left[\frac{\omega - \omega_0}{2} T\right]. \tag{16}$$

Answer:

$$\begin{aligned}
x(\omega) &= A \int_0^T \mathrm{e}^{-\mathrm{i}\omega_0 t} \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}t \\
&= A \int_0^T \mathrm{e}^{\mathrm{i}(\omega - \omega_0)t}\, \mathrm{d}t \\
&= A \frac{\mathrm{e}^{\mathrm{i}(\omega-\omega_0)t}}{\mathrm{i}(\omega - \omega_0)} \Big|_0^T \\
&= -\frac{\mathrm{i}A}{\omega - \omega_0} \left[\mathrm{e}^{\mathrm{i}(\omega-\omega_0)T} - 1\right] \\
x^*(\omega) &= \frac{\mathrm{i}A}{\omega - \omega_0} \left[\mathrm{e}^{-\mathrm{i}(\omega-\omega_0)T} - 1\right].
\end{aligned}$$

So then

$$\begin{aligned}
P(\omega) &= |x(\omega)|^2 = x^*(\omega) x(\omega) \\
&= \frac{A^2}{(\omega - \omega_0)^2} \left[1 + 1 - \left(\mathrm{e}^{\mathrm{i}(\omega-\omega_0)T} + \mathrm{e}^{-\mathrm{i}(\omega-\omega_0)T}\right)\right] \\
&= \frac{2A^2}{(\omega - \omega_0)^2} \left[1 - \cos\left((\omega - \omega_0)T\right)\right].
\end{aligned}$$

Alternatively, one may have factored out an exponential in a preceding step to arrive at a different Fourier transform:

$$\begin{aligned}
x(\omega) &= -\frac{\mathrm{i}A}{\omega - \omega_0} \left[\mathrm{e}^{\mathrm{i}(\omega-\omega_0)T} - 1\right] \\
&= -\frac{\mathrm{i}A}{\omega - \omega_0} \mathrm{e}^{\frac{\mathrm{i}}{2}(\omega-\omega_0)T} \left[\mathrm{e}^{\frac{\mathrm{i}}{2}(\omega-\omega_0)T} - \mathrm{e}^{-\frac{\mathrm{i}}{2}(\omega-\omega_0)T}\right] \\
&= \frac{2A}{\omega - \omega_0} \mathrm{e}^{\frac{\mathrm{i}}{2}(\omega-\omega_0)T} \sin\left(\frac{\omega - \omega_0}{2} T\right) \\
x^*(\omega) &= \frac{2A}{\omega - \omega_0} \mathrm{e}^{-\frac{\mathrm{i}}{2}(\omega-\omega_0)T} \sin\left(\frac{\omega - \omega_0}{2} T\right)
\end{aligned}$$

Then

$$\begin{aligned}
P(\omega) &= |x(\omega)|^2 = x^*(\omega) x(\omega) \\
&= \frac{4A^2}{(\omega - \omega_0)^2} \sin^2\left(\frac{\omega - \omega_0}{2} T\right) \\
&= A^2 T^2 \left[\frac{\sin\left(\frac{\omega-\omega_0}{2} T\right)}{\frac{\omega-\omega_0}{2} T}\right]^2 \\
&= A^2 T^2 \operatorname{sinc}^2 \left[\frac{\omega - \omega_0}{2} T\right].
\end{aligned}$$

(d) Sketch the power spectrum under the assumption that $T \gg 1/\nu_0 \gg (\omega_0/2\pi)^{-1}$. When you make your figure, do it as detailed as possible. At the very least, focus on the value and shape near the resonance $\omega = \omega_0$. To get the peak amplitude, it will help to use techniques similar to those in Problem 2. After that, find the first zeros of the spectrum. Label what you need to.

Answer: Near resonance $\omega \approx \omega_0$, we can Taylor expand the cosine function around 0:

$$\begin{aligned} P(\omega \approx \omega_0) &\approx \frac{2A^2}{(\omega - \omega_0)^2}\left[1 - \left(1 - \frac{1}{2}(\omega - \omega_0)^2 T^2\right)\right] \\ &\approx A^2 T^2. \end{aligned}$$

The first 0s of the power will occur then the cosine function has an argument of $(\omega - \omega_0)T = \pm 2\pi$, or, when $\omega = (\pm 2\pi + \omega_0 T)/T = \omega_0 \pm 2\pi/T$. This is when the cosine term goes to 0, but the denominator does not (unlike at resonance).