Cume #408 - Solutions
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You are living comfortably on planet Proxima Centauri b. A small spacecraft crashed into your planet recently that contained, among other things, an astronomy textbook that describes the fourth nearest star to you and its planets, written by the intelligent (arguably) inhabitants of one of them. You learn that the beautiful G2V star relatively nearby actually has a diverse solar system. Your job is to try to detect signatures of those planets and confirm the textbook using a full array of instrumentation available to you. For simplicity, assume circular orbits for Earth, Jupiter, and you, and that you view the distant solar system edge on (inclination of 90°).

The *anticipated passing grade* is 75%, or 63 out of 84.5 total points (77 + 7.5).

Show all work clearly and please write legibly, and if you can't solve something completely, at least give an idea of how you might go about it. Make sure you are careful to answer ALL parts of each question. Don't spend too much time in the beginning on one question, move on and try them all and then come back if you need to. DO NOT use your calculators for any formulae or constants, only to calculate. Start each numbered problem on a new piece of paper. Only write on the front of each piece of paper, please. Take your time, think clearly, read each sentence carefully, ask for clarification, and best of luck to you!

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1. (32 points). You are going to try to measure the **radial velocity** of that distant Sun.

   (a) First, what are Kepler's 3 laws? Feel free to present these in the context of the (our) Solar System, and don't worry about numbering or ordering them if you don't want to. (7 points)

   Answer:

   i. planets orbit in ellipses with the Sun at one focus,
   ii. the line joining the planet and Sun sweeps out equal areas in equal times,
   iii. the square of a planet's period is proportional to the cube of its semi-major axis,

   or, precisely

   $$P^2 = \frac{4\pi^2}{G(M_\odot + M_p)} a^3. \tag{1}$$

   $P^2 = a^3$ would be acceptable for our solar system, given the right units.

   (b) Describe briefly what the radial velocity of a star is (1), a simple technique with which one might measure the radial velocity of a star (2), and then how obtaining such a measurement(s) might tell you about the properties of a planetary system (3). (6 points)

   Answer: (1) The radial velocity of a star is its projected motion along the line-of-sight of the observer. (2) One could measure the radial velocity by taking spectra and using the Doppler shift of spectral lines over time. [One could use interferometers too but that is slightly more complicated ...] (3) One would look for periodic motion of the star's radial velocity about the system's barycenter which gives information about planetary orbits such as eccentricity and orbital period. Without inclination you cannot get mass. If the period and type of star is known, one could get an estimate of stellar mass and hence an estimate of semimajor axis.

   (c) Assuming for the moment you are in a reference frame at rest, what is the maximal radial-velocity signature of the Sun you expect to measure due only to the motion of the Earth? Due only to the motion of Jupiter? Provide your answers in $\mathrm{m\,s^{-1}}$. (6 points)

   Answer: In general, for any planet p orbiting the Sun,

   $$m_p v_p = m_\odot v_\odot, \tag{2}$$

   so that

   $$v_\odot = \frac{m_p}{m_\odot} v_p = \frac{m_p}{m_\odot} \frac{2\pi a_p}{P_p}, \tag{3}$$

when the planet velocity can be estimated as $v_p = 2\pi a_p/P_p$, where a is the semimajor axis and P the orbital period. One could also arrive at this by computing the center of mass $m_{\odot}x = m_p(a_p - x)$. Using the appropriate values we then find

$$
\begin{aligned}
v_{\odot,\oplus} &= 0.09\,\mathrm{m\,s^{-1}}, && (4)\\
v_{\odot,\mathrm{J}} &= 12.8\,\mathrm{m\,s^{-1}}. && (5)
\end{aligned}
$$

(d) Now that you know what type of radial velocities to expect, you can design an observational plan. Say you want to observe the Sun only in the Hα line. Considering only the Sun and these two planets, at what wavelength would you expect to measure the solar Hα line at its largest departure from the rest wavelength? (If you don't think you got reasonable values in part (c), guesstimate something here and use those velocities). (5 points)

Answer: When the tug from Earth and Jupiter are in phase we would find

$$
\begin{aligned}
\frac{\Delta\lambda}{\lambda_0} &= \frac{v_{\odot,\oplus} + v_{\odot,\mathrm{J}}}{c}, && (6)\\
\Delta\lambda &= 2.8235 \times 10^{-5}\,\mathrm{nm}. && (7)
\end{aligned}
$$

Therefore one would expect to find the Hα line at $656.9 \pm 2.8235 \times 10^{-5}\,\mathrm{nm}$.

(e) What spectral resolution would be required for your spectrograph to precisely measure the difference between the rest line profile and shifted line profile? (2 points)

Answer: It would be

$$\frac{\lambda}{\Delta\lambda} = \frac{v_{\mathrm{RV},\odot}}{c} = 23,000,000. \tag{8}$$

(f) Spectrographs are not available with this type of spectral resolution. Explain briefly, then, how you would modify your observations and/or analysis to have a better chance of measuring the RV signature of the Sun due to Earth + Jupiter (ignoring all other systematics and sources of noise). (2 points)

Answer: You would not just use 1 line, but try to measure a Doppler shift over many, many lines.

(g) In real observations you need to undertake, what other "sources" of a radial-velocity signal could contribute to your measurement, either periodic or non periodic. Name or describe 2 other sources. Perhaps drawing a picture would help you identify some of them (though not required). (4 points)

Answer:

i. There is the relative motion between the Sun and your planet, due to a combination of the Sun's relative motion to Proxima Centauri and the orbit of Proxima Centauri b about its host. These are non periodic and periodic, respectively. The relative velocity between the stars is $\approx -20\,\mathrm{km\,s^{-1}}$ and given the orbital parameters of the planet, its velocity is about $\approx 50\,\mathrm{km\,s^{-1}}$. These are typical values. Earth orbits the Sun at a bit over half that speed.

ii. The Sun's gravitational redshift adds a component of about $636\,\mathrm{m\,s^{-1}}$. This is non periodic, and for Hα would amount to $\Delta\lambda = 0.0014$.

iii. Convection on the Sun causes a "blueshift" of absorption lines in low-resolution spectra on the order of $-0.5\,\mathrm{km\,s^{-1}}$. Non periodic.

iv. Sunspots can cause periodic amplitudes at a few $\mathrm{m\,s^{-1}}$.

v. Lump everything else into "instrumental noise."

2. (23 points). Now we are going to look at transits of the Sun to see what we can learn of the planets.

Answer:

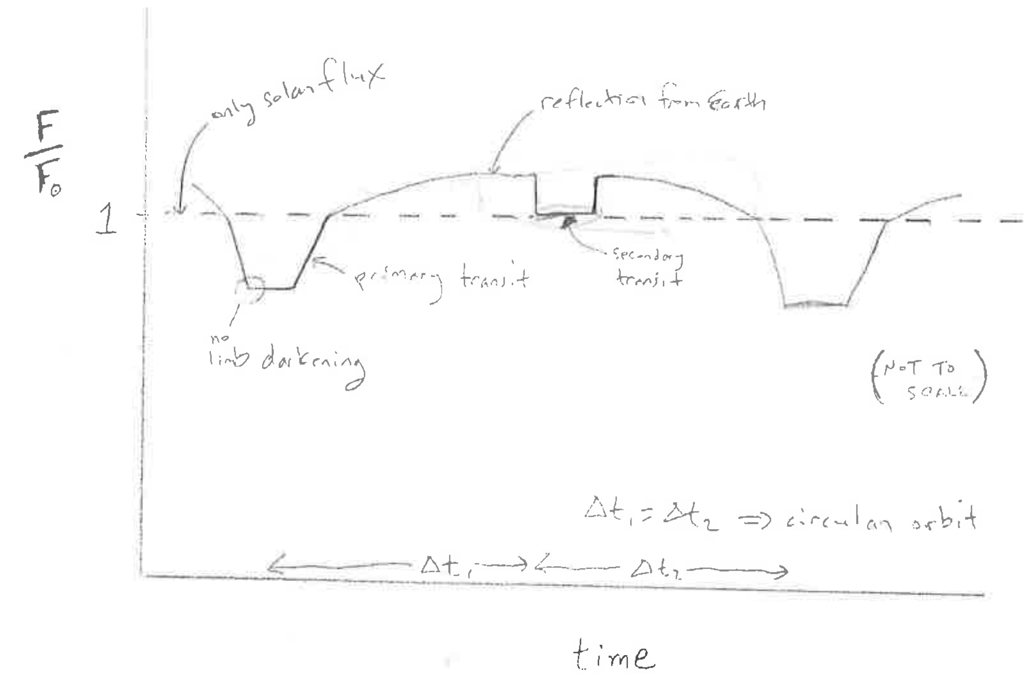

Figure 1: Representative light curve for 2(a).

(a) Draw a curve showing the solar light you would observe photometrically (in the broadband visible) as Earth goes through a full period around the Sun (note: Earth's albedo is non negligible, but assume for simplicity that Earth does not **emit** any light in your bandpass). Your y-axis should be $F(t)/F_\odot$ (where $F(t)$ is the total flux you measure at each time t) and the x-axis is time. This should be a **qualitative curve only**, you don't need to put values on the y axis except for $F/F_\odot = 1$, which you should use as a reference. The time axis is one full period. Make sure your light curve is clear enough so that it's obvious when/if the curve is below/above 1. Label any features of your curve that show something happening. Ignore solar limb darkening for now (but see below). (8 points)

Answer: Figure 1 shows what is needed. Reflected light implies the existence of a secondary transit, so a higher flux. Ignore thermal radiation. The primary should appear a bit steeper, boxier.

(b) If you were now to be quantitative, what would be the numerical value of the minimum of the light curve, $\Delta F/F_\odot$, for the Earth passing in front of the Sun? For Jupiter? I.e., what is the minimum at the center of the primary transits? Don't redraw, just compute. Hint: Luminosity. (5 points)

Answer: Using the luminosity and flux interchangeably, the flux right when the primary transit starts is $F_\odot \sim R_\odot^2 T_\odot^4$. At the bottom of the transit its the solar flux minus the flux blocked out by a small circle of radius R_p. Therefore

$$\frac{\Delta F}{F_\odot} = \frac{F_\odot - (F_\odot - 4\pi\sigma R_p^2 T_\odot^4)}{F_\odot} \approx \left(\frac{R_p}{R_\odot}\right)^2. \qquad (9)$$

For Earth, the primary transit would be $\Delta F/F \approx 8.4 \times 10^{-5}$. Micromag. For Jupiter, 0.01. 10 millimags.

(c) What would change in your drawing if Earth's orbit were sufficiently eccentric? (2 points)

Answer: If there was eccentricity, the primary and secondary transits would not be evenly spaced in time. Also, the width of the primary and secondary transits would change, although this is a very small effect compared to the first.

(d) What is stellar limb darkening? (2 points).

Answer: It is the decreasing intensity from the center of the stellar disk to the limb when observed in visible light. It is due to the fact that the light coming from the limb of the star/Sun originates from a higher level (i.e., cooler) in the atmosphere than light from the center of the disk.

(e) How would the figure you drew earlier change if we took limb darkening of the Sun into account? You can explain briefly in words and/or a drawing if necessary. (2 points)

Answer: Limb darkening would cause the edges to not be sharp at the bottom of the primary transits. It would cause the slope of the ingress egress to be shallower, and it overall makes the transit very U-shaped.

(f) What would be the approximate transit time you would expect to observe (width of primary transit in time) across the Sun for Earth? For Jupiter? (4 points)

Answer: This is basically the distance traveled divided by the speed of transit.

$$t = \frac{2R_\odot}{2\pi a}P = \frac{1}{\pi}\frac{R_\odot}{a}P. \tag{10}$$

$t_\oplus = 13$ hr and $t_J = 28$ hr for Earth and Jupiter, respectively.

3. (16 points). Let's look at potentially directly imaging those distant planets around the Sun.

(a) Firstly, what is the approximate parallax angle (or more appropriately, annual parallax) of the Sun from your Proxima Centauri b (in arcsec)? Reminder: the annual parallax for Earth dwellers is half the observed angle through which a distant star appears to shift in half an Earth year. A drawing might help if you need it. (4 points)

Answer: You can't use $p = 1/d$, as that is derived for Earth's orbital parameters. The parallax angle p in the small-angle formula is $2pd = 2a$ where d is the distance between stars and a is the semi-major axis. The distance is about 4.2 ly. Computing p and converting from radians to arcseconds gives $p \approx 0.389''$.

$$p = \frac{a}{d} = \frac{0.05\,\text{AU}}{4.2\,\text{ly}} = 1.89 \times 10^{-7}\,\text{rad} = 1.08 \times 10^{-5}\,\text{deg} = 0.0389''. \tag{11}$$

(b) What is the maximum angular separation (in arcseconds) between the Sun and Earth from your perspective? The Sun and Jupiter? (4 points)

Answer: The angular separation using the small angle formula is $\theta = a/d$. We find $\theta_\oplus = 0.77''$ and $\theta_J = 4.07''$.

(c) With a 3.5m telescope at your disposal (like the one at APO), what would be the diffraction limit of the telescope somewhere in the visible spectrum? (3 points)

Answer: The diffraction limit is given by $\theta_{DL} = 1.22\lambda/D$ (radians). In this case, using $\lambda = 500$ nm, we find $\theta_{DL} = 0.04''$.

(d) Could you then, in principle, resolve the separation of the Sun and Earth and the Sun and Jupiter? Assume the Sun's light can be sufficiently dimmed with a coronagraph if that were necessary. (2 points)

Answer: Yes, in principle both planets could be resolved.

(e) If the atmosphere at your alien observatory is similar to the atmosphere at APO, would you realistically be able to resolve Earth and the Sun using the same telescope ? What about Jupiter and the Sun? Explain. Assume you haven't developed adaptive optics yet (morons). (3 points)

Answer: The best seeing is on the order of $1''$, so it's unlikely Earth could be resolved. But Jupiter should still be possible.

4. (6 points). Let's say you've successfully detected solar system planets, confirming the textbook, and now you want to move on and discover more exoplanets near other stars. Besides the 3 methods you've already used here, name and very, very briefly describe **two** other techniques you could use to potentially detect the other planets.

Answer:

(a) You could try microlensing, staring at distant sources photometrically to look for characteristic brightenings of a foreground star passing in between you and the source and forming a "lens." If a planet is orbiting, a magnification feature will arise at some point in the crossing.

(b) Astrometry to measure the transverse motion of a star due to planet tugs.

(c) Transit-timing observations. If one transiting exoplanet is found, small variations in its orbital period due to other planets can result.

Things needed:

- $G = 6.674 \times 10^{-8}\,\mathrm{cm^3\,g^{-1}\,s^{-2}}$
- $h = 4.135668 \times 10^{-15}\,\mathrm{eV\,s}$
- $a_{\mathrm{Prox.Cent.b}} = 0.05$ AU; Semi-major axis of Proxima Centauri b
- $R_\oplus = 6371$ km
- $m_\oplus = 6 \times 10^{27}$ g
- $a_\oplus = 1$ AU
- $P_\oplus = 1$ yr
- $R_{\mathrm{J}} = 70000$ km
- $m_{\mathrm{J}} = 1.9 \times 10^{30}$ g
- $a_{\mathrm{J}} = 5.2$ AU
- $P_{\mathrm{J}} = 11.86$ yr
- $m_\odot = 1.99 \times 10^{33}$ g
- $R_\odot = 6.96 \times 10^5$ km
- $P_{\mathrm{Prox.Cent.b}} = 11.2$ day; Orbital period of Proxima Centauri b
- $D_{\mathrm{Prox.Cent.}} = 4.2$ ly; Distance to Proxima Centauri
- $M_* = 0.123\,\mathrm{M_\odot}$; Mass of Proxima Centauri (an M6 red dwarf)
- $\lambda_{\mathrm{H}\alpha} = 656.9$ nm; Rest wavelength of Hα $\leftarrow$ obtained if using $13.6\,\mathrm{eV}\left(\frac{1}{2^2} - \frac{1}{3^2}\right)$
- $1\ \mathrm{AU} = 1.5 \times 10^{13}$ cm;