Cume #391 - Solutions
MATHS!!
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This exam deals with some basic math that can occur frequently in astrophysics research. These topics cover things that aren't always trivial to just "look up" to find the right answer. The first few are to get your feet wet and then they pick up a little in complexity. **Of the first 4 questions, you only need to do 3 of them.** If you do 4, only your top 3 scores will count, but make sure you allow enough time to get to the final two questions. The *anticipated passing grade* is 75%.

Show all work clearly and please write legibly, and if you can't solve something completely, at least give an idea of how you might go about it. Make sure you are careful to answer ALL parts of each question. Dont spend too much time in the beginning on one question, move on and try them all and then come back if you need to. DO NOT use your calculators for any formulae or constants, only to calculate. Start each numbered problem on a new piece of paper. Take your time, think clearly, read each sentence carefully, ask for clarification, and best of luck to you!

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1. (10 points). Consider the two vectors $\boldsymbol{a} = \hat{\imath} + 3\hat{\jmath} - \hat{k}$ and $\boldsymbol{b} = 2\hat{\imath} + \hat{\jmath} + \hat{k}$.

   (a) Determine a **unit vector** perpendicular to the plane of these two vectors.

   Answer: The cross product of these two vectors gives a perpendicular one:

   $$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} i & j & k \\ 1 & 3 & -1 \\ 2 & 1 & 1 \end{vmatrix} = 4\hat{\imath} - 3\hat{\jmath} - 5\hat{k}$$

   . Since we want a unit vector, the length is $\sqrt{50}$, so we divide everything by that.

   (b) What is the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, in degrees?

   Answer: Let's use the dot product

   $$\boldsymbol{a} \cdot \boldsymbol{b} = ab\cos\theta = 4.$$
   $$\cos\theta = \frac{4}{\sqrt{11 \cdot 6}}, \longrightarrow \theta = 60.5^\circ.$$

2. (10 points). Consider the matrix

   $$M = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

   which satisfies the equation

   $$M\boldsymbol{x} = \lambda\boldsymbol{x}, \tag{1}$$

   where λ are the eigenvalues and x are the eigenvectors in a Cartesian space $\boldsymbol{x} = (x, y, z)$.

   (a) What is the determinant of M?

   Answer: The determinant is

   $$0\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + 0\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0.$$

(b) What are the eigenvalues λ of Eq. (1)?

Answer: The secular equation is

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0$$

which means

$$-\lambda(\lambda^2 - 1) = 0.$$

The roots are thus $\lambda = -1, 0, 1$.

(c) What are the corresponding normalized eigenvectors such that $|\boldsymbol{x}| = 1$?

Answer: The eigenvalue equation Eq. (1) is rewritten as

$$\begin{pmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

For $\lambda = -1$, we have

$$x + y = 0, \qquad z = 0.$$

We can write this as $\boldsymbol{x}_1 = (1, -1, 0)$, but since the vector length is $\sqrt{2}$, the normalized eigenvector is

$$\boldsymbol{x}_1 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right).$$

For $\lambda = 0$,

$$y = 0, \qquad x = 0,$$

so

$$\boldsymbol{x}_2 = (0, 0, 1).$$

For $\lambda = 1$,

$$-x + y = 0, \qquad z = 0,$$

so

$$\boldsymbol{x}_3 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right).$$

3. **(10 points).** Integration. Compute the indefinite integral of

$$\int e^x \sin x \, \mathrm{d}x. \tag{2}$$

Answer: Integrate by parts.

$$\begin{aligned} u &= \sin x, & \mathrm{d}v &= e^x \\ \mathrm{d}u &= \cos x \, \mathrm{d}x, & v &= e^x \end{aligned}$$

Now

$$\int e^x \sin x \, \mathrm{d}x = e^x \sin x - \int e^x \cos x \, \mathrm{d}x.$$

$$\begin{aligned} u &= \cos x, & \mathrm{d}v &= e^x \\ \mathrm{d}u &= -\sin x \, \mathrm{d}x, & v &= e^x \end{aligned}$$

$$\int e^x \sin x \, \mathrm{d}x = e^x \sin x - e^x \cos x - \int e^x \sin x \, \mathrm{d}x.$$

Collecting terms gives

$$\int e^x \sin x \, \mathrm{d}x = \frac{1}{2} e^x (\sin x - \cos x). \tag{3}$$

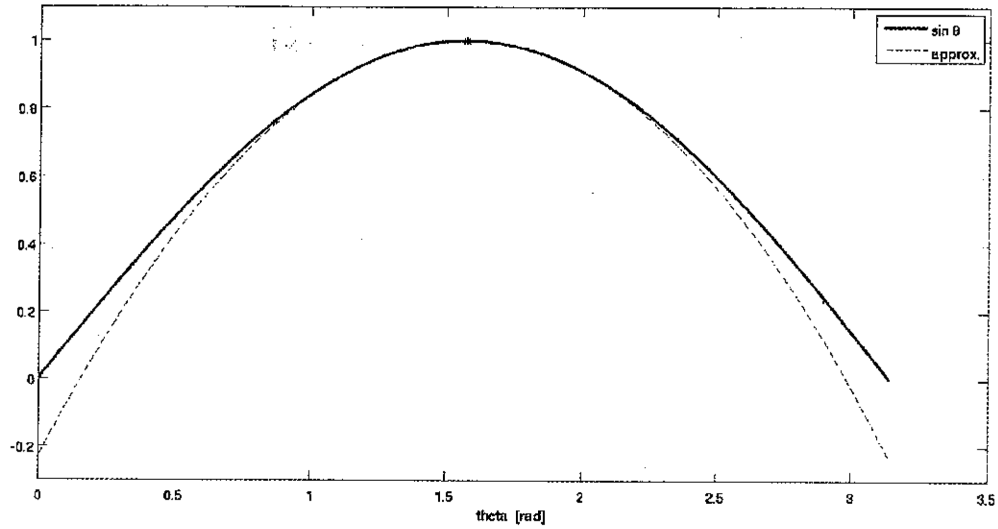

Figure 1: Solution to Problem 4. It shows $y = \sin\theta$ as well as the 2nd-order approximation at $\pi/2$.

4. (10 points). Taylor expansion.

   (a) Taylor expand $f(\theta) = \sin\theta$ around the value $\theta = \pi/2$ to 2nd order to find an approximate expression near that value.

   Answer: The expansion is

$$f(\theta) \approx f(a) + f'(a)(\theta - a) + \frac{1}{2}f''(a)(\theta - a)^2 + \dots \tag{4}$$

   Then

$$\sin\theta \approx 1 + 0 - \frac{1}{2}\left(\theta - \frac{\pi}{2}\right)^2 + \dots$$

   (b) Sketch $\sin\theta$ and its approximation from 0 to π.

   Answer: Figure 1 shows the function as well as the approximation at the chosen value.

5. (40 points). Solving differential equations analytically and numerically.
   Consider the continuity equation for a steady fluid system

$$\nabla \cdot (\rho \boldsymbol{v}) = 0, \tag{5}$$

   where ρ is density and $\boldsymbol{v}$ is the fluid velocity. Just consider a one-dimensional system where:

$$\nabla \rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\hat{\boldsymbol{x}}, \tag{6}$$
$$\rho \rightarrow \rho(x), \tag{7}$$
$$\boldsymbol{v} \rightarrow v\hat{\boldsymbol{x}}. \tag{8}$$

   (a) (6 points). Describe in 1-2 sentences what this equation describes physically. Then expand out the equation using the operators in one dimension.

   Answer: The continuity equation describes the conservation of mass flux, which says that the mass flux divergence is zero. Mass coming in equals mass going out. Expanding in 1D gives

$$\rho\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}\rho}{\mathrm{d}x} = 0. \tag{9}$$

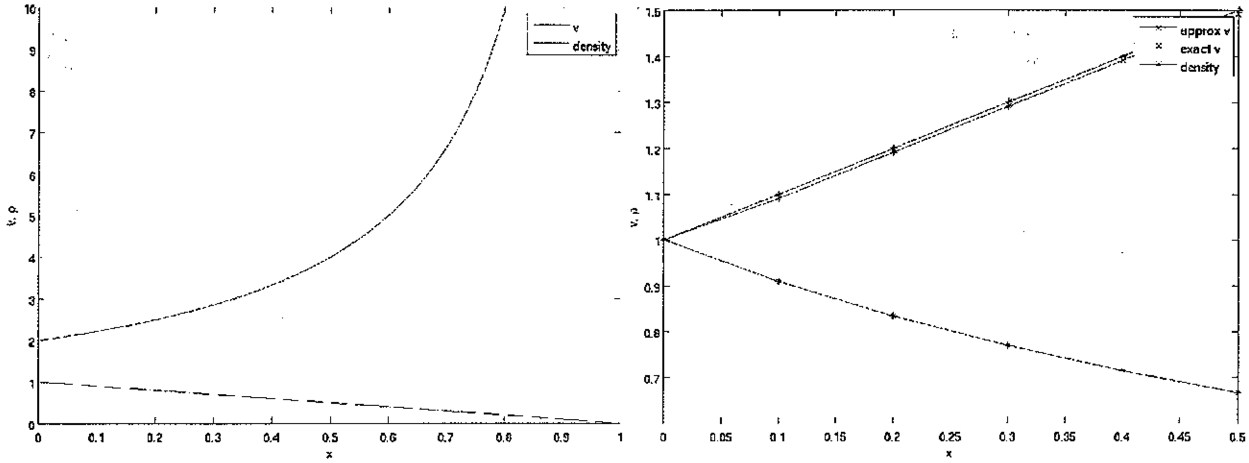

Figure 2: Solutions for Problem 5. The left panel is the solution for part (c), and the right is the solution for part (e). Included is the approximate Euler solution as well as the exact solution for the velocity.

(b) (4 points). What is the *degree* and what is the *order* of this equation? What type of equation is it (mathematically speaking)?

Answer: This is an ordinary, first-order, linear differential equation. It's homogeneous too.

(c) (10 points). Assume you know that the density varies as $\rho(x) = 1 - x$, and that you have a boundary condition that $v(x = 0) = 2$. Derive the analytic solution for $v(x)$. Sketch $\rho(x)$ and $v(x)$ on the same axis from $(0, 1)$.

Answer: One can rewrite Eq. (9) separating variables as

$$\frac{\mathrm{d}v}{v} = -\frac{\rho'}{\rho}\mathrm{d}x. \tag{10}$$

Since $\rho' = -1$, we have

$$\begin{aligned} \int \frac{\mathrm{d}v}{v} &= \int \frac{1}{1-x}\mathrm{d}x, \\ \ln v &= -\ln(1-x) + C, \\ v &= \frac{C}{1-x}, \\ v &= \frac{2}{1-x}, \end{aligned}$$

where the last line uses the boundary condition. In general, it is easier starting from Eq. (10) to just write

$$\begin{aligned} \frac{\mathrm{d}v}{v} &= -\frac{\mathrm{d}\rho}{\rho}, \\ \ln v &= -\ln\rho + C, \\ v &= C\rho^{-1}. \end{aligned}$$

The velocity and density profiles are shown in Fig. 2 on the left.

(d) (10 points). Now let's assume we don't know the density stratification analytically, only numerically, yet still want to know the velocity structure that satisfies continuity! Sketch out the simplest way possible to do this algorithmically, keeping in mind the form of the equation in general is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y(x)). \tag{11}$$

Assume again that you also know an *initial value* of y.

Answer: The simplest way is an Euler method or something similar. You don't know y, but you know that it's derivative is

$$y' = \frac{y(x+h) - y(x)}{h} = f(x, y(x)), \tag{12}$$

where h is some step size in $x_i = x_{i-1} + h$. If we discretize, then

$$y(x_{i+1}) = y(x_i) + hf(x_i, y(x_i)). \tag{13}$$

In this specific example,

$$\begin{aligned} y &\equiv v, & (14)\\ f &\equiv -\frac{\rho'}{\rho}v. & (15) \end{aligned}$$

(e) (10 points). Given the following table of values and the initial condition $v(0) = 1$, find a solution for a solution $v(x_i)$ to four decimal points for all 6 grid points. Plot the density and the final result for velocity on the same axis ($\rho' = \mathrm{d}\rho/\mathrm{d}x$).

x	0.0	0.1	0.2	0.3	0.4	0.5
ρ	1.0	0.9091	0.8333	0.7692	0.7143	0.6667
ρ'	-0.9091	-0.8333	-0.6993	-0.5952	-0.5128	-0.4762

Answer: Here, $i = 1 \ldots 6$ and $\mathrm{d}x = 0.1$. The values in the table actually represent $\rho = (1+x)^{-1}$. We will use

$$v(x_{i+1}) = v(x_i) - \mathrm{d}x\frac{\rho'(x_i)}{\rho(x_i)}v(x_i). \tag{16}$$

So then

$$\begin{aligned} v(x_1) = v(0.0) &= 1.0000\\ v(x_2) = v(0.1) = 1.0000 - 0.1\frac{-0.9091}{1.0}(1.0000) &= 1.0909\\ v(x_3) = v(0.2) = 1.0909 - 0.1\frac{-0.8333}{0.9091}(1.0909) &= 1.1909\\ v(x_4) = v(0.3) = 1.1909 - 0.1\frac{-0.6993}{0.8333}(1.1909) &= 1.2908\\ v(x_5) = v(0.4) = 1.2908 - 0.1\frac{-0.5952}{0.7692}(1.2908) &= 1.3907\\ v(x_6) = v(0.5) = 1.3907 - 0.1\frac{-0.5128}{0.7143}(1.3907) &= 1.4905. \end{aligned}$$

This is plotted in Fig. 2 along with the exact solution. Since the density profile is

$$\rho(x) = (1+x)^{-1}, \tag{17}$$

the velocity

$$v(x) = 1 + x, \tag{18}$$

when plugged in with the given initial value.

6. (30 points). Convolution.

Recall the *convolution theorem*, which states that the convolution of two functions in real space is equivalent to the inverse Fourier transform of their product in Fourier space. In the time domain in 1D, for example, the convolution C of two functions is

$$C(t) = (f * g)(t) = \int_{-\infty}^{\infty} f(t')g(t-t')\,\mathrm{d}t'. \tag{19}$$

The theorem suggests that the convolution can be computed as

$$f * g = \mathcal{F}^{-1}\left\{\mathcal{F}\{f\} \cdot \mathcal{F}\{g\}\right\}, \tag{20}$$

where $\mathcal{F}\{\}$ and $\mathcal{F}^{-1}\{\}$ denote general Fourier and inverse Fourier transform operations, respectively. Let's actually define the Fourier transform and inverse transform for some function y, as well as the Dirac delta function as

$$\begin{aligned}
y(\omega) &= \int_{-\infty}^{\infty} y(t)e^{\mathrm{i}\omega t}\,\mathrm{d}t,\\
y(t) &= \int_{-\infty}^{\infty} y(\omega)e^{-\mathrm{i}\omega t}\,\mathrm{d}\omega,\\
\delta(\omega) &= \int_{-\infty}^{\infty} e^{-\mathrm{i}\omega t}\,\mathrm{d}t.
\end{aligned}$$

(a) (4 points). Describe in a few words what is a convolution? An example or two might be helpful.

Answer: A convolution of two functions provides a new function that expresses the integrated overlap between the two as one of them is shifted. It is an output signal after and input gets altered by some internal mechanism.

(b) (3 points). What might be an advantage of using Eq. (20) as opposed to Eq. (19)?

Answer: In real space the computational need is of order $O(N^2)$ operations for vectors of length N. If one utilizes an FFT, the Fourier definition results in about $O(N\log N)$ operations. In higher dimensions it becomes even more advantageous.

(c) (6 points). Prove that the theorem holds using the definitions given for general f and g.

Answer: Using Eq. (19) to get to Eq. (20), we can plug in the definitions to get

$$\begin{aligned}
C(t) &= \int\!\!\int\!\!\int f(\omega)e^{-\mathrm{i}\omega t'}g(\omega')e^{-\mathrm{i}\omega'(t-t')}\,\mathrm{d}t'\,\mathrm{d}\omega\,\mathrm{d}\omega',\\
&= \int\!\!\int f(\omega)g(\omega')e^{-\mathrm{i}\omega' t}\int e^{-\mathrm{i}(\omega-\omega')t'}\,\mathrm{d}t'\,\mathrm{d}\omega\,\mathrm{d}\omega',\\
&= \int\!\!\int f(\omega)g(\omega')e^{-\mathrm{i}\omega' t}\delta(\omega-\omega')\,\mathrm{d}\omega\,\mathrm{d}\omega',\\
&= \int f(\omega)g(\omega)e^{-\mathrm{i}\omega t}\,\mathrm{d}\omega,
\end{aligned}$$

where the last line is clearly the inverse FT operation of a product of functions in Fourier space. If one instead starts from Eq. (20) and tries to work backwards to Eq. (19), this is not a proof really, but you'd find

$$\begin{aligned}
C(t) = (f*g)(t) &= \mathcal{F}^{-1}\{\mathcal{F}\{f\}\cdot\mathcal{F}\{g\}\},\\
&= \mathcal{F}^{-1}\left\{\int f(t')e^{\mathrm{i}\omega t'}\,\mathrm{d}t'\int g(t'')e^{\mathrm{i}\omega t''}\,\mathrm{d}t''\right\},\\
&= \int e^{-\mathrm{i}\omega t}\,\mathrm{d}\omega\int f(t')e^{\mathrm{i}\omega t'}\,\mathrm{d}t'\int g(t'')e^{\mathrm{i}\omega t''}\,\mathrm{d}t'',\\
&= \int\!\!\int\!\!\int \mathrm{d}t'\mathrm{d}t''\mathrm{d}\omega\, f(t')g(t'')e^{\mathrm{i}\omega(-t+t'+t'')},\\
&= \int\!\!\int \mathrm{d}t'\mathrm{d}t''\, f(t')g(t'')\,\delta(-t+t'+t''),\quad t''\to t-t',\\
&= \int f(t')g(t-t')\,\mathrm{d}t'.
\end{aligned}$$

(d) (12 points). Consider the convolution of a boxcar function and a gaussian function along the infinite x axis. Let

$$\begin{aligned}
f(x) &= \begin{cases}1, & -1<x<1\\ 0, & \text{otherwise}\end{cases}\\
g(x) &= e^{-ax^2}.
\end{aligned}$$

Use the theorem to try to find an expression for the convolution of these two functions. Get as far as you can. (*Hint: First compute separately the transforms. Upon evaluating the FT of the* *gaussian, consider computing* $\mathrm{d}/\mathrm{d}\omega$ *(or whatever you call your frequency variable) of the entire integral to help you evaluate it!)*

Answer: Let k be the spatial frequency. Doing the gaussian first,

$$g(k) = \int e^{-ax^2} e^{ikx}\, \mathrm{d}x = \int e^{-ax^2} (\cos kx + \mathrm{i} \sin kx)\, \mathrm{d}x.$$

The term with sin vanishes because it's antisymmetric. Call the remaining integral I. Then using the hint

$$\begin{aligned}
\frac{\mathrm{d}I}{\mathrm{d}k} &= -\int e^{-ax^2} x \sin kx\, \mathrm{d}x, \\
&= \frac{1}{2a} \int \sin kx\, \mathrm{d}(e^{-ax^2}), \\
u = \sin kx \qquad & \mathrm{d}v = \mathrm{d}(e^{-ax^2}), \\
\mathrm{d}u = k \cos kx, \mathrm{d}x, \qquad & v = e^{-ax^2}, \\
&= \frac{1}{2a} \left[\Big|_{-\infty}^{\infty} e^{-ax^2} \sin kx - \int e^{-ax^2} k \cos kx\, \mathrm{d}x \right], \\
&= -\frac{k}{2a} \int e^{-ax^2} \cos kx\, \mathrm{d}x, \\
\frac{\mathrm{d}I}{\mathrm{d}k} &= -\frac{k}{2a} I, \\
I = g(k) &= e^{-k^2/4a},
\end{aligned}$$

where in the last line we just solved the differential equation. This shows that the transform of a gaussian is another gaussian. For the boxcar

$$\begin{aligned}
f(k) &= \int_{-\infty}^{\infty} f(x) e^{ikx}\, \mathrm{d}x = \int_{-1}^{1} e^{ikx}\, \mathrm{d}x, \\
&= \frac{1}{ik} e^{ikx} \Big|_{-1}^{1}, \\
&= \frac{1}{ik} \left(e^{ik} - e^{-ik} \right), \\
f(k) &= \frac{2 \sin k}{k} = 2\mathrm{sinc}(k).
\end{aligned}$$

Now that we've computed the FT of both functions, we can plug them into our convolution theorem Eq. (20) to get

$$C(x) = 2 \int e^{-k^2/4a} \frac{\sin k}{k} e^{-ikx}\, \mathrm{d}k. \tag{21}$$

No idea how to do that. Good enough for here.

(e) (5 points). Even if you don't get to the right answer, sketch f, g, and what you think the result of this convolution would be for a given a. Describe what is happening.

Answer: The functions and convolution are shown in Fig. 3 for $a = 0.2$. The gaussian basically smooths out the sharp edges of the boxcar function.

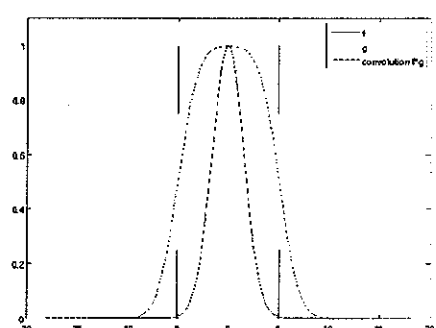

Figure 3: The functions f and g from Problem 6 part (d). The gaussian used a value of $a = 0.2$.