Cume #385 - Solutions
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January 25, 2014

This exam deals with basis astrophysics knowledge. The problems are rather straightforward, and the goal here is for you to be accurate and quick in your calculations. The *anticipated passing grade* is 80%, or 64 points out of 80 points.

Show all work clearly and please write legibly, and if you can't solve something completely, at least give an idea of how you might go about it. Make sure you are careful to answer ALL parts of each question. Dont spend too much time in the beginning on one question, move on and try them all and then come back if you need to. DO NOT use your calculators for any formulae or constants, only to calculate. Start each numbered problem on a new piece of paper. Take your time, think clearly, read each sentence carefully, ask for clarification, and best of luck to you!

Some numbers you may need:

$$\begin{aligned} h &= 4.135668 \times 10^{-15}\,\mathrm{eV\,s} \\ 1\,\mathrm{Ly} &= 9.46 \times 10^{15}\,\mathrm{m} \\ 1\,\mathrm{AU} &= 1.496 \times 10^{13}\,\mathrm{cm} \\ G &= 6.674 \times 10^{-8}\,\mathrm{cm^3\,g^{-1}\,s^{-2}} \end{aligned}$$

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1. (22 points). Astrometry.

   (a) The Hipparcos Mission measured parallax angles to about $0.001''$. Approximately how far (in km) from a dime would you need to be to observe it subtending such an angle?

   Answer: Use small-angle formula. Estimate a dime to be about 2cm.

$$\begin{aligned} s &= r\theta \longrightarrow r = s\theta^{-1} && (1) \\ s &= 2\,\mathrm{cm} && (2) \\ \theta &= 0.001'' \times \frac{1^\circ}{3600''} \times \frac{\pi}{180} = 4.85 \times 10^{-9}\,\mathrm{rad} && (3) \\ r &= \left(2 \times 10^{-5}\,\mathrm{km}\right)\left(4.85 \times 10^{-9}\right)^{-1} = 4120\,\mathrm{km}. && (4) \end{aligned}$$

   (b) Barnard's star (Bernie's star?) has a proper motion of $\mu = 10.3577''\,\mathrm{yr}^{-1}$ and a parallax angle of $p = 0.54901''$. It's Hα line as measured from Earth is $\lambda = 656.034\,\mathrm{nm}$.
   Describe in one sentence the Hα line, including what "series" it's from. Compute an approximate value for its laboratory wavelength (use this value in any subsequent calculation when needed).

   Answer: The Hydrogen-alpha transition is from the Balmer series and describes an electron transitioning (in emission) from the $n = 3$ to $n = 2$ energy levels in a hydrogen atom. Its energy change is

$$\Delta E = 13.6\,\mathrm{eV}\left(\frac{1}{2^2} - \frac{1}{3^2}\right) = 1.8889\,\mathrm{eV}. \quad (5)$$

   The wavelength is obtained from

$$\begin{aligned} \Delta E &= h\nu = \frac{hc}{\lambda} && (6) \\ \lambda &= \frac{hc}{\Delta E} && (7) \\ \lambda &= \left(4.135668 \times 10^{-15}\,\mathrm{eV\,s}\right)\left(3 \times 10^{10}\,\mathrm{cm\,s^{-1}}\right)\left(1.8889\,\mathrm{eV}\right) = 6568.41\,\text{Å}. && (8) \end{aligned}$$

   This value is off by the accepted one by about 6 angstroms because of the approximate speed-of-light value, but we'll use it moving forward anyway.

(c) What is Barnard's star's radial velocity (in $\mathrm{km\,s^{-1}}$)?

Answer: The radial velocity along the line of sight can be measured from the Doppler shift of the Hα line.

$$v_r = c\frac{\lambda_{obs} - \lambda_{ref}}{\lambda_{ref}} = 3 \times 10^5\,\mathrm{km\,s^{-1}} \left(\frac{656.034 - 656.841}{656.841}\right) = -368.6\,\mathrm{km\,s^{-1}}. \tag{9}$$

This is a bit high, if one uses a more precise value for c and the resulting reference wavelength, the velocity is $v_r = -113\,\mathrm{km\,s^{-1}}$.

(d) What is the transverse velocity (in $\mathrm{km\,s^{-1}}$) of Barnard's star?

Answer: This is just related to the proper motion if distance is known. The distance to Barnard's star is

$$r = p^{-1} = 1.8215\,\mathrm{pc} = 1.8215 \times 3.26 = 5.938\,\mathrm{Ly} = 5.617 \times 10^{13}\,\mathrm{km}. \tag{10}$$

Then

$$v_T \equiv \frac{ds}{dt} = r\frac{d\theta}{dt} \equiv r\mu = (5.617 \times 10^{13}\,\mathrm{km}) \left(\frac{10.3577''}{1\,\mathrm{yr}} \times \frac{1^\circ}{3600''} \times \frac{\pi}{180} \times \frac{1\,yr}{3.15 \times 10^7\,\mathrm{s}}\right) = 89.54\,\mathrm{km\,s^{-1}}. \tag{11}$$

(e) What is the speed of Barnard's star and in what direction is it moving through space with respect to the line-of-sight of Earth?

Answer: The star's speed is

$$v = \sqrt{v_r^2 + v_T^2} = 379.3\,\mathrm{km\,s^{-1}}, \tag{12}$$

if using the values above, or $144\,\mathrm{km\,s^{-1}}$, if using the smaller value of v_r. Since the radial velocity is negative, the star is coming towards Earth. It's angle with respect to the line of sight is $\theta = \arctan(v_T/v_r) = 13^\circ$ if using the larger radial velocity or $\theta = 39^\circ$ if using the smaller value.

2. (14 points). Radiation.

Recall this version of Planck's function for blackbody radiation:

$$B_\lambda(T) = \frac{2hc^2}{\lambda^5}\frac{1}{\exp\left(\frac{hc}{\lambda kT}\right) - 1}. \tag{13}$$

(a) Derive an approximate expression of Planck's function **to first order**, valid at long wavelengths.

Answer: At long wavelengths, $\lambda \gg hc/kT$ and the exponential can be expanded as $e^x \approx 1 + x$. We then have

$$B_\lambda \approx \frac{2ckT}{\lambda^4}. \tag{14}$$

(b) What happens to the expression you just derived as $\lambda \to 0$? What is the historical name of this "problem" at short wavelengths?

Answer: As the wavelength goes to zero, the amount of radiation goes to infinity. This is known as the "ultraviolet catastrophe."

(c) Derive an approximate expression of Planck's function **to first order**, valid at short wavelengths.

Answer: We can rewrite the expression as

$$B_\lambda = \frac{2hc^2}{\lambda^5}\left(\exp\left(\frac{hc}{\lambda kT}\right) - 1\right)^{-1} = \frac{2hc^2}{\lambda^5}\exp\left(-\frac{hc}{\lambda kT}\right)\left(1 - \frac{1}{\exp\left(\frac{hc}{\lambda kT}\right)}\right)^{-1}. \tag{15}$$

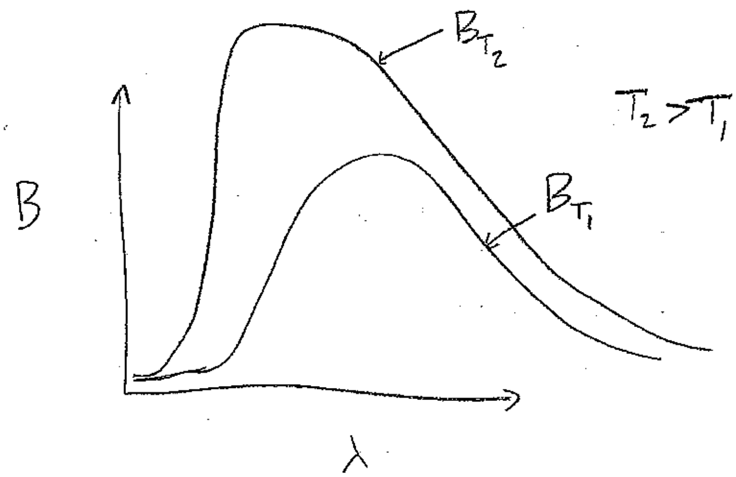

Figure 1: Figure for blackbody problem.

Now we basically have $x(1-x)^{-1}$, with x small. To first order, using the binomial theorem this gives $x(1-x)^{-1} \approx x(1+x+\ldots) \approx x$. We thus have

$$B_\lambda \approx \frac{2hc^2}{\lambda^5} \exp\left(-\frac{hc}{\lambda kT}\right). \tag{16}$$

(d) Very roughly sketch $B_T(\lambda)$ vs. λ from your result in (c) for finite λ and for two different temperatures T_1 and T_2, where $T_2 > T_1$. No need for numbers on your axes here, just sketch the qualitative behavior of the two curves and label appropriately.

Answer: See Figure 1. Higher intensity at all wavelengths and bluer peak for the hotter object.

3. (22 points). Orbits.

(a) Comet Hale-Bopp has an orbit about the Sun with eccentricity $e = 0.9951$ and a semi-major axis $a = 186.5\,\mathrm{AU}$. If its last perihelion occurred in 1997, was its previous perihelion before or after the death of J.C.?

Answer: The period

$$P = a^{3/2} = 186.5^{3/2} = 2547\,\mathrm{yr}. \tag{17}$$

Its previous perihelion would have been before J.C.

(b) Does it get closer to the Sun than Earth does? Where is it at its furthest distance from the Sun with respect to the Kuiper Belt and the Oort Cloud?

Answer: The closest distance it gets to the Sun is $r_\mathrm{p} = a(1-e) = 0.914$AU, which is inside Earth's orbit. Its furthers distance is $r_\mathrm{a} = a(1+e) = 372.1$AU. The Kuiper Belt extends out to only about 50AU. The Oort Cloud may extend out to tens of thousands of AU. Therefore the comet is somewhere between these areas at aphelion.

(c) Compute the mass of the Sun (approximately) from comet Hale-Bopp's orbital parameters.

Answer: Kepler's 3rd Law is

$$P^2 = \frac{4\pi^2}{G} \frac{a^3}{M_\odot + M_\mathrm{comet}}. \tag{18}$$

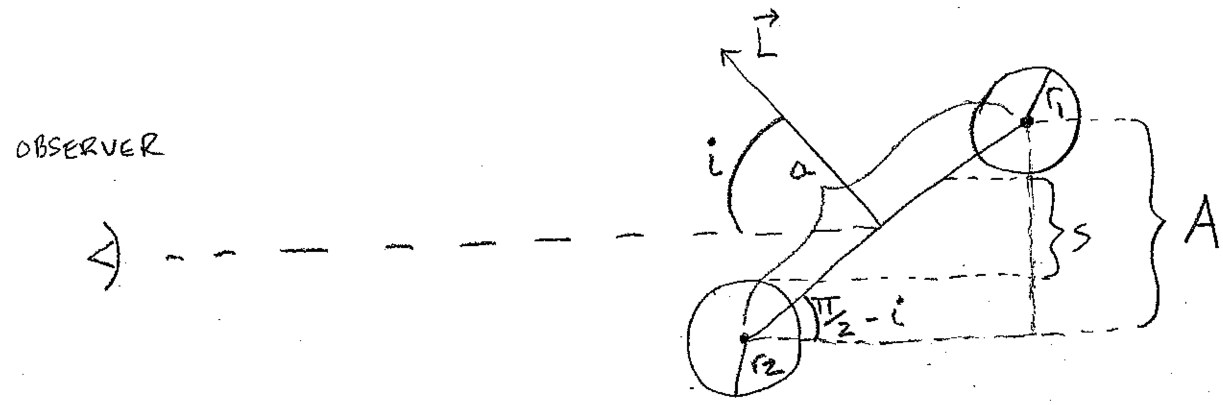

Figure 2: Figure for eclipse problem.

We can ignore the mass of the comet and find

$$M_\odot \simeq \frac{4\pi^2}{6.674\times10^{-8}\,\mathrm{cm^3\,g^{-1}\,s^{-2}}}\frac{\left(2.79\times10^{15}\,\mathrm{cm}\right)^3}{(8.1\times10^{10}\,\mathrm{s})^2} = 1.96\times10^{33}\,\mathrm{g}, \tag{19}$$

which isn't too far off.

(d) Assume 2 stars in a binary system are in circular orbits about their center of mass, separated by a distance a. The angle of inclination (angle between your line of sight and the angular momentum vector of the orbit) is i. The stellar radii are r_1 and r_2.

Consider this binary system oriented such that it is just *non-eclipsing.* Make a side-view sketch showing the observer, the orbital plane, and circles representing the stars. Draw the angular momentum vector and label the inclination angle, as well as any other information you know.

Answer: See Figure 2. Both stars are along the line-of-sight of the observer at this moment so you can use the separation distance correctly.

(e) Use your figure to find an expression for the smallest angle i that will just barely produce an eclipse, in terms of the stellar radii and separation distance.

Answer: In general, the inclination angle is related to the distances by

$$A = a\sin(\pi/2 - i) = s + r_1 + r_2 = a\cos i, \tag{20}$$

or

$$\cos i = \frac{s + r_1 + r_2}{a}. \tag{21}$$

For $s = 0$, the expression is simply

$$\cos i = \frac{r_1 + r_2}{a}. \tag{22}$$

(f) If $r_1 = 10\,R_\odot$, $r_2 = 1\,R_\odot$, and $a = 2\mathrm{AU}$, what is the minimum value of i that will result in an eclipse?

Answer: 2AU is approximately $430\,R_\odot$, so

$$i = \cos^{-1}\left(\frac{11\,R_\odot}{430\,R_\odot}\right) \approx 88.5^\circ. \tag{23}$$

4. (22 points). Rotation.

(a) Estimate the minimum rotation period for a typical white dwarf star. *Hint: consider the break-up speed.*

Answer: The fastest rotation occurs right before a star would break up, when the centripetal force on some surface fluid element would equal the force due to gravity.

$$\frac{GMm}{R^2} = \frac{mv^2}{R} \longrightarrow v = \sqrt{\frac{GM}{R}}. \tag{24}$$

A "typical" white dwarf could have a mass of $1M_\odot$ (although most are much less massive, but anything less than 1.4 will do) and a radius about Earth's, $R \approx 6.4 \times 10^8$ cm, or $\approx 0.01R_\odot$. Thus

$$v = \left(\frac{(6.674 \times 10^{-8}\,\mathrm{cm^3\,g^{-1}\,s^{-2}})\,(1.99 \times 10^{33}\,\mathrm{g})}{6.4 \times 10^8\,\mathrm{cm}}\right)^{1/2} = 4.56 \times 10^8\,\mathrm{cm\,s^{-1}}. \tag{25}$$

The period is therefore

$$P = \frac{2\pi R}{v} = \frac{2\pi \cdot 6.4 \times 10^8\,\mathrm{cm}}{4.56 \times 10^8\,\mathrm{cm\,s^{-1}}} = 8.8\,\mathrm{s}. \tag{26}$$

(b) Estimate the minimum rotation period for a typical neutron star.

Answer: Let the neutron star have a mass of $1.4M_\odot$ and a radius of 10km. Then.

$$v = \left(\frac{(6.674 \times 10^{-8}\,\mathrm{cm^3\,g^{-1}\,s^{-2}})\,(1.4)(1.99 \times 10^{33}\,\mathrm{g})}{10^6\,\mathrm{cm}}\right)^{1/2} = 1.36 \times 10^{10}\,\mathrm{cm\,s^{-1}}. \tag{27}$$

The period is therefore

$$P = \frac{2\pi R}{v} = \frac{2\pi \cdot 10^6\,\mathrm{cm}}{1.36 \times 10^{10}\,\mathrm{cm\,s^{-1}}} = 4.61 \times 10^{-4}\,\mathrm{s}. \tag{28}$$

(c) How would the period you found in (b) compare to the rotational period of the Sun if it collapsed into an object the size of a neutron star (conserving angular momentum and assuming no mass loss)? *Hint: Assume constant density and remember that the moment of inertia I of a solid sphere can be obtained from*

$$\mathrm{d}I = \frac{2}{3}r^2\rho\,\mathrm{d}V, \tag{29}$$

where $\mathrm{d}V$ *is a volume element.*

Answer: Conservation of angular momentum requires $L_{\text{before}} = L_{\text{after}}$, where $L = I\omega$ with moment of inertia I. This can be calculated as (using tools from ASTR 565)

$$\begin{aligned}
\mathrm{d}I &= \frac{2}{3}r^2\rho\,\mathrm{d}V, && (30)\\
\mathrm{d}I &= \frac{2}{3}r^2\mathrm{d}m, \qquad \text{using } \rho = \frac{\mathrm{dm}}{\mathrm{dV}} && (31)\\
\mathrm{d}I &= \frac{8\pi}{3}r^4\rho\,\mathrm{d}r, \qquad \text{using } \mathrm{dm} = 4\pi \mathrm{r}^2\rho\,\mathrm{dr} && (32)\\
I &= \frac{8\pi}{3}\rho\int_0^R r^4\,\mathrm{d}r = \frac{8\pi}{15}\rho R^5, \qquad \text{since } \rho \text{ is constant} && (33)\\
I &= \frac{2}{5}MR^2. && (34)
\end{aligned}$$

The expression for the angular momentum using $\omega = 2\pi/P$ is

$$L = I\omega = \frac{4\pi}{5}\frac{MR^2}{P}. \tag{35}$$

Then

$$\begin{aligned}
L_\odot &= L_{\text{ns}}. && (36)\\
\frac{4\pi}{5}\frac{M_\odot R_\odot^2}{P_\odot} &= \frac{4\pi}{5}\frac{M_\odot R_{\text{ns}}^2}{P_{\text{ns}}}, && (37)\\
P_{\text{ns}} &= P_\odot\left(\frac{R_{\text{ns}}}{R_\odot}\right)^2. && (38)
\end{aligned}$$

Taking $P_\odot \approx 27\,\mathrm{day} \approx 2.33 \times 10^6\,\mathrm{s}$ we have

$$P_{\rm ns} \approx 2.33 \times 10^6\,\mathrm{s} \left(\frac{10^6\,\mathrm{cm}}{6.96 \times 10^{10}\,\mathrm{cm}} \right)^2 \approx 4.8 \times 10^{-4}\,\mathrm{s}, \tag{39}$$

which is slightly larger than the break-up speed of the neutron star of 1.4 solar masses. If the solar rotation period was taken to be 25 day instead, the answer would be smaller. In any case, the answer should be similar.

Cume #385 - Points Distribution

1. 22 points
   - (a) 3 points:
     1 point for reasonable estimate of dime size
     2 points for right formula and answer
   - (b) 7 points:
     3 points for knowing Balmer and 2,3
     2 points for knowing 13.6 eV and $1/n^2$
     1 points for correct formula for λ
     1 points for correct answer
   - (c) 3 points:
     2 points for right formula
     1 point for finding right answer, including sign
   - (d) 6 points:
     2 points for getting distance
     2 points for right use of small-angle formula
     2 points for finding right answer
   - (e) 3 points:
     1 points for correct formula
     2 points for finding right answer and direction

2. 14 points
   - (a) 3 points
   - (b) 2 points
   - (c) 4 points
   - (d) 5 points:
     1 point for overall shape right
     2 points for the hotter with more flux at all λ
     2 points for hotter with bluer peak

3. 22 points
   - (a) 3 points
   - (b) 5 points:
     2 points for right distances
     3 points for answers of both questions correctly
   - (c) 4 points:
     2 points for right formula
     2 points for decent answer, or a note about a really bad answer
   - (d) 5 points for figure
   - (e) 3 points
   - (f) 2 points including a good $R\odot$

4. 22 points
   - (a) 8 points:
     4 points for centrip=grav and right v
     2 points for estimates of mass and radius
     2 points for decent period
   - (b) 4 points for good mass and good radius
   - (c) 10 points:
     2 points for $L = I\omega$
     3 points for computation of I
     3 points for setting up cons. equation (even if wrong equation, units should)
     1 points for correct solar rotation
     1 points for good answer