Cume # 373 – Solutions

Jason Jackiewicz

October 13, 2012

This exam covers material related to the article "*Rapid Orbital Decay in the 12.75-Minute Binary White Dwarf J0651+2844*," Hermes et al., ApJL, **757** (2012). The *anticipated* passing grade is 70%, or **45 points** out of a total of **65 points**.

Show all work clearly and please write **legibly**, and if you can't solve something completely, at least give an idea of how you might go about it. Make sure you are careful to answer **ALL parts** of each question. Don't spend too much time in the beginning on one question, move on and try them all and then come back if you need to. **DO NOT use your calculators for any formulae or constants, only to calculate.** Start each numbered problem on a new piece of paper. Take your time, think clearly, *read each sentence carefully*, ask for clarification, and best of luck to you!

A few things you may or may not need:

- Solar radius: $R_\odot = 6.96 \times 10^{10}\,\mathrm{cm}$
- Solar mass: $M_\odot = 2.0 \times 10^{33}$ g
- Stefan-Boltzmann constant: $\sigma = 5.68 \times 10^{-5}\,\mathrm{erg\,cm^{-2}\,s^{-1}\,K^{-4}}$
- Gravitational constant: $G = 6.67 \times 10^{-8}\,\mathrm{cm^3\,g^{-1}\,s^{-2}}$

I Stellar properties (20 points)

1. **[10 points]. (a) Discuss succinctly what a white dwarf (WD) star is and how it is created. Provide sufficient details of a WD's past evolutionary tale and its current state, using properties like masses, time scales, initial conditions, composition, size, etc. (b) How are WDs identified observationally? (c) What different types of WDs are there? (d) What are the various things that can occur when a binary system contains a WD? (e) Sketch a (relatively accurate) H-R diagram with these two stars on it, including tracing out their approximate evolutionary tracks to their current positions (label as much as you can regarding the WDs along their path).**

ANSWER: Start with a solar mass star (but stars up to about $8\,M_\odot$ can become WDs). It exhausts its H core and leaves the main sequence. H begins burning in the shell and begins moving up the sub-giant and then red-giant branch. It begins to fuse He in the (degenerate) core, it may go through a He flash, and settles on the horizontal branch, like the main sequence but burning He, not H. It begins to move up the AGB where it can be $100\,R_\odot$, and begins to lose mass. After a runaway mass loss where all outer layers are gone (a significant amount of mass loss can occur), what's left is a hot carbon core with thin shells of H and He, which gets hotter and hotter while maintaining most of its luminosity and it runs across the H-R diagram in a few tens of thousands of years. A planetary nebula can form from the expelled atmosphere. In about 10^5 years, it burns all of its surface fuel and is thus mostly carbon and maybe half its original mass, shrinking to the size of the Earth ($0.01\,R_\odot$), continuously cooling while not changing size any more (degenerate).

Although they are mostly carbon/oxygen in the interior, the thin outer atmospheres dominate their spectrum, showing either H absorption lines (DA) or He (DB). In addition, the absorption features are extremely broadened due to the strong pressure near the star's surface. No real metallic lines are seen in WDs as these heavier nuclei are gravitationally forced below the surface. Can look for them in planetary nebulae. WDs have been observed to pulsate in g modes.

If a WD is in a binary, mass transfer can occur and the WD shrinks and a dwarf nova, classical nova, or a type I supernova can occur. The WD primary in this paper is very low mass. The maximum mass a WD can have is $1.44\,M_\odot$.

A rough H-R diagram is shown in Fig. 1. The primary star's luminosity is 16 times solar, and the smaller star's is about 0.35 solar.

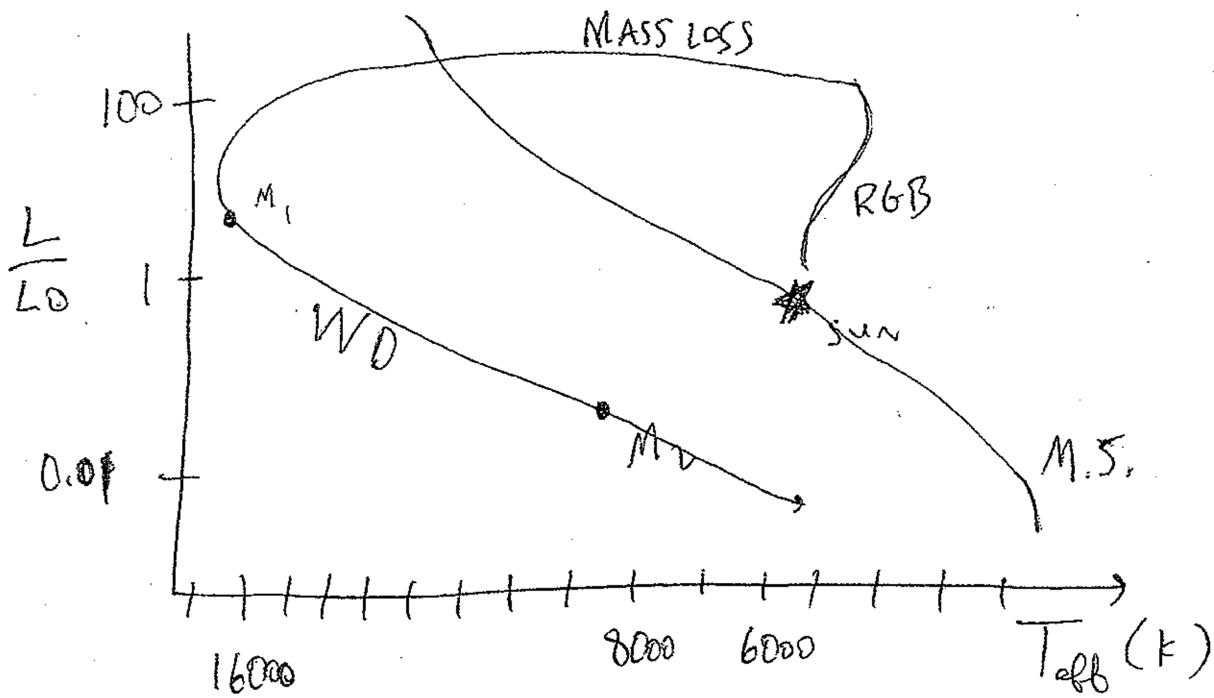

Figure 1: White dwarf evolution.

2. **[10 points]. (a) Calculate the distance to the binary system in the paper. (b) Then using this value, or by other means, calculate the smaller star's apparent magnitude. (c) Finally, confirm that the less massive star contributes (approximately) the small percentage of the total light in the g-band as the paper indicates. Consider only the g-band, and ignore any limb-darkening computations. *Hint: Remember that a difference of 5 apparent magnitudes of two stars corresponds to a difference of 100 in brightness.***

ANSWER: The hint suggests that the flux $F = L/4\pi d^2$ or brightness of two stars are related by

$$\frac{F_2}{F_1} = 100^{(m_1-m_2)/5}. \tag{1}$$

The paper states a dereddened apparent magnitude for the primary of $g_0 = 19.1 = m_1$. Its absolute magnitude in the g-band is given as $M_1 = 8.9$. From the flux ratio expression, we can replace the parameters of one star with its parameters at 10 parsec:

$$\frac{F_{10}}{F_1} = 100^{(m_1-M_1)/5} = \left(\frac{d}{10}\right)^2, \text{ in parsec.} \tag{2}$$

Solving for d gives

$$d = 10^{(m_1-M_1+5)/5} \approx 1097\,\text{parsec.} \tag{3}$$

Carrying out the same procedure for the secondary where $M_2 = 12.5$ and using this distance:

$$m_2 = 5\log d - 5 + M_2 = m_1 - M_1 + M_2 = 22.7. \tag{4}$$

Note that the difference in absolute and apparent magnitudes are equal.

With absolute magnitudes of $M_1 = 8.9$ and $M_2 = 12.5$, the flux ratio is

$$\frac{F_2}{F_1} = 100^{(M_1-M_2)/5} = 0.0363 \approx 3.7\%, \tag{5}$$

as mentioned in the paper on page 3 before section 4.

II Light-curve analysis (26 points)

3. [12 points]. See Figure 3 (the figure in the exam, not in the paper). (a) On this figure, first label each eclipse with their correct names, and from the parameters given in the paper, describe the orbital configuration of the stars for each eclipse (i.e., show how the geometry corresponds to the associated dips in the light curve). It's probably easiest to make your arguments using luminosity. Numbers here aren't necessary. (b) Then, show that the luminosity ratio does not equal the flux (brightness) ratio (the one you found before), and give a reason why this might be. (c) Finally, using the flux ratios you've already computed and any stellar parameters you need from the article, show that you can approximately reproduce the values for both of the eclipse depths in Figure 3 (again, ignore limb darkening). You can work in flux or magnitudes, if you choose magnitudes, you should therefore refer to Figure 2 of the paper.

ANSWER: The deep eclipse (primary) is the small star passing in front of the primary, and the secondary eclipse is the small star passing behind the primary. This is because $R_1 > R_2$, $T_1 > T_2$, *and thus* $L_1 > L_2$. *Let's use luminosity for a proxy of brightness for now (but below shows this isn't exact, even though their distances are equal). Say that the total out of eclipse flux is* $L_1 + L_2$. *When the small star passes in from of the bigger one, the flux is* $L_1 + L_2 - R_2^2 T_1^4 = L_1 + R_2^2(T_2^4 - T_1^4) = L_1 - \epsilon$, *since* T_1 *is larger. When the small star passes behind, the total flux is just from the primary* L_1. *So the deeper (primary) transit will be when the small, cool WD passes in front of the hot WD. Just because the primary star is more luminous does not guarantee this configuration. If the secondary star were slightly hotter than the primary star, but still less luminous (still small), the deeper transit would be when the secondary passed behind the primary. The flattish part at the bottom of the secondary eclipse is because the small star is behind the larger star for a sufficient amount of time.*

One would expect the flux and luminosity ratios to be equal since the pair of stars are at the same distance. The flux ratio of the two stars is 3.7%, but the luminosity ratio $R_2^2 T_2^4 / R_1^2 T_1^4$ *is only about 1.1%. This indicates that one or both of these stars deviate from a blackbody spectrum, as has been suspected for cool WDs for a while.*

We saw from before that $F_2/F_1 = 0.037$. *Consider the total flux is* $F_1 + F_2 = 1$. *For the secondary (smaller) eclipse, the only star we see is the primary star, so the depth should correspond to* $F_1/(F_1 + F_2) - 1 = 1/1.037 - 1 = -0.0357$, *consistent with the figure when working in flux. If working in magnitudes as in the figures in the paper, one would instead compute*

$$\Delta m = -2.5 \log \frac{F_1}{F_1 + F_2} = 0.0394, \tag{6}$$

which is consistent with the plot given in the paper (figure 2).

For the primary (larger) eclipse we saw above that the depth should correspond to the sum of the fluxes minus the small fraction of the large star that is blocked $(R_2/R_1)^2 F_1$. *So in flux, we find that the depth given by*

$$\frac{F_1 + F_2 - (R_2/R_1)^2 F_1}{F_1 + F_2} - 1 = -0.14, \tag{7}$$

in approximate accordance with the figure. Working instead in magnitudes, we find $\Delta m =$ 0.165, *in approximate agreement with the figures in the paper.*

4. [**6 points**]. (a) **What is Doppler "boosting" (or "beaming") and identify on the light curve where you see it? (b) Identify the ellipsoidal variations in the light curve too and explain what these are. For each case, explain any symmetries or antisymmetries of these two phenomena. (c) How do you know from the light curve that the orbit is not highly eccentric? Please use Figure 3 in the exam to label and identify these things.** ***Hint: If you're stuck, think about the possible things that can happen when 2 stars orbit close to each other and are observed photometrically in narrow bandpasses.***

   ANSWER: The Doppler boosting is manifest in the small asymmetry of the amplitude maxima between eclipses. This is due to the stars being partially red or blue-shifted out of the instrument bandpass, thus changing the total amount of light at different phases. At 0.25 phase, you have one star blue- and the other one redshifted, and at 0.75 phase you have the opposite. Since the stars give off different amounts of light, the peak will be different at these two phases.

   The ellipsoidal variations are due to tidal forces on the stars from close-in interactions. This is the symmetric sinusoidal pattern between eclipses. This effects changes the shape of the stars to non-spherical so that there is maximal light at elongation (more surface area), where the stars are elongated in the perpendicular (to the line-of-sight) direction.

   If the orbit were eccentric, the eclipses would not be space equidistantly in phase.

5. [8 points]. With the questions below we will think about and derive the $O - C$ equation given in section 4 of the paper.

(a) First, in your own words, explain why such an equation is necessary and what it represents.

(b) We're going to define some quantities that differ slightly from the article. Let the observed mid-point times be

$$O \equiv T^{\rm obs}(E) = T_0 + P(E)[E(t) - E_0], \tag{8}$$

where E_0 is the first epoch and $T_0 \equiv T(E_0)$ is the mid-point time of the first epoch. Note the implicit time dependence of the period (through E: $E = t/P$). Also let

$$C \equiv T^{\rm mod}(E) = T_0 + P_0(E(t) - E_0) \tag{9}$$

be the modeled mid-point times assuming a constant period fixed at P_0 (P_0 defined here is different from the text, which says it's the observed period at the first eclipse.).

Draw on a $T - E$ plot what a few typical observations may look like (O) and what the model might predict (C) over a few epochs if the period is slowly changing. Label the plot with as much detail as possible, as this will help you answer the next part.

(c) Use a Taylor expansion to derive the $O - C$ equation, by expanding the observed times *to second order* around some epoch E'. Assume that any $E \gg 2E'$.

ANSWER:

(a) This relation describes how to measure the period of eclipses that is changing with time.

(b) A straight line for the model and a scatter of points for the observations. The difference in T (Δy) is the period and the difference in E (Δx) is the successive epochs.

(c)

$$\begin{aligned}
T^{\rm obs}(E) &= T_0 + P(E - E_0) && (10)\\
T^{\rm obs}(E) &= T(E') + \left.\frac{\mathrm{d}T}{\mathrm{d}E}\right|_{E=E'}(E - E') + \frac{1}{2}\left.\frac{\mathrm{d}^2T}{\mathrm{d}E^2}\right|_{E-E'}(E - E')^2 + \cdots && (11)\\
\left.\frac{\mathrm{d}T}{\mathrm{d}E}\right|_{E=E'} &= P(E') && (12)\\
\left.\frac{\mathrm{d}^2T}{\mathrm{d}E^2}\right|_{E-E'} &= \left.\frac{\mathrm{d}P}{\mathrm{d}E}\right|_{E-E'} = \left.\frac{\mathrm{d}P}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}E}\right|_{E-E'} = P(E')\frac{\mathrm{d}P}{\mathrm{d}t} = P(E')\dot{P} && (13)\\
E &\gg 2E' && (14)\\
E^2 - 2EE' + E'^2 &\approx E^2 && (15)\\
O &\equiv T^{\rm obs} = T(E') + P(E')E + \frac{1}{2}P'\dot{P}E^2 + \cdots && (16)\\
C &\equiv T^{\rm mod} = T_0 + P_0E && (17)\\
O - C &= \Delta T' + \Delta P'E + \frac{1}{2}P'\dot{P}E^2 && (18)\\
\Delta T' &= T' - T_0 && (19)\\
\Delta P' &= P' - P_0 && (20)
\end{aligned}$$

III General relativity (19 points)

6. [4 points]. **Explain briefly what gravitational waves are, in what astronomical situations they may be expected to be found, how they can be detected, and if you're familiar with any current or future experiments intended to directly measure them.**

   ANSWER: Gravitational waves are mostly analogous to electromagnetic waves and are thought to transport the energy of gravitational radiation. They create a propagating distortion in space-time that can be detected by large interferometers that get displaced when a wave impinges on them. These waves are thought to emanate from strong gravitational sources, such as binary neutron or WD stars.

7. [15 points]. Compute the period decay rate predicted from general relativity and compare with the observed value quoted in the paper in the discussion section. You can do this! For simplicity, consider two bodies attracting each other according to Newton's laws and orbiting in circular orbits. (It's probably best not to plug in any numbers until part (e)).

   (a) Draw and label an accurate, simple figure of the orbits of the two stars studied in the paper (as if you were looking down at them from above, perpendicular to the orbital plane). This will help you with answering the rest of the questions.
   (b) Show that the total energy in the system (kinetic + potential) of the two bodies is $E = -Gm_1m_2/2r$, where $r = r_1 + r_2$ is the separation between the stars. (Make sure that you keep in mind the stars are orbiting the center of mass, as your figure in (a) should show.)
   (c) Now use the energy expression to find the formula for the stars' rate of approach, i.e., the rate of change of the orbital separation r: $\mathrm{d}r/\mathrm{d}t$ (assuming constant masses). Use the fact that the energy lost to gravitational waves over time is given in general relativity by

$$-\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{32G^4m_1^2m_2^2(m_1+m_2)}{5c^5r^5}, \tag{21}$$

   where c is the speed of light, m_i are stellar masses, and G is the gravitational constant.
   (d) Write down the expression for how the time rate of change of the period $\mathrm{d}P/\mathrm{d}t$ is related to time rate of change of the separation $\mathrm{d}r/\mathrm{d}t$.
   (e) Finally, use your expression in (d) to compute the time rate of change of the orbital period and compare to the value quoted in the paper.

ANSWER:

(a) See Figure 2 for geometry.

(b) The kinetic energy is

$$\begin{aligned}
T &= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 && (22)\\
v_1 &= \omega r_1 = \frac{2\pi}{P}r_1 && (23)\\
v_1^2 &= \frac{4\pi^2}{P^2}r_1^2 = \frac{G(m_1+m_2)}{r^3}r_1^2 && (24)\\
T &= \frac{1}{2}\frac{G(m_1+m_2)}{r^3}r_1^2m_1 + \frac{1}{2}\frac{G(m_1+m_2)}{r^3}r_2^2m_2 && (25)\\
r &= r_1 + r_2,\ \ m_1r_1 = m_2r_2 && (26)\\
r_1 &= \frac{m_2}{m_1+m_2}r,\ \ r_2 = \frac{m_1}{m_1+m_2}r && (27)\\
T &= \frac{G}{2r}\left[\frac{m_1m_2^2}{m_1+m_2} + \frac{m_1^2m_2}{m_1+m_2}\right] = \frac{Gm_1m_2}{2r} && (28)
\end{aligned}$$

The potential energy is $V = -Gm_1m_2/r$, so that the total energy of the system is $E = T + V = -Gm_1m_2/2r$. If the Virial theorem ($T = -V/2$) is used by first showing that $V = -Gm_1m_2/r$ and then that $E = T + V = V/2 = -Gm_1m_2/2r$, that's fine too.

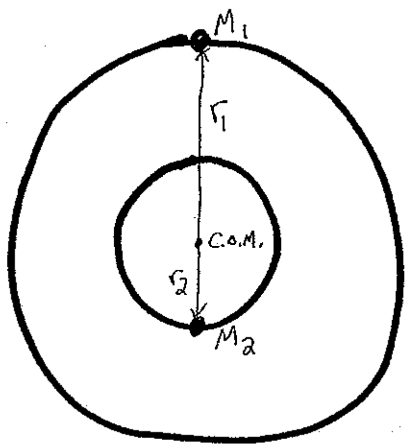

Figure 2: Orbital configuration (assuming circles) for these 2 stars orbiting the c.o.m.

(c) *Solving the energy equation for r and finding its time rate of change is*

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{2r^2}{Gm_1m_2}\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{64G^3m_1m_2(m_1+m_2)}{5c^5r^3}. \tag{29}$$

(d) *The shrinking of the radius is related to the change in period using Kepler's 3rd law:*

$$\dot{P} = \frac{\mathrm{d}P}{\mathrm{d}t} = \frac{6\pi^2r^2}{PG(m_1+m_2)}\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{6(64)\pi^2G^2}{P\,5c^2}\frac{m_1m_2}{r}. \tag{30}$$

Or (as some will write it)

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{3\pi r^{1/2}}{\sqrt{G(m_1+m_2)}}\frac{\mathrm{d}r}{\mathrm{d}t}. \tag{31}$$

r needs to be found from Kepler's 3rd law:

$$r = \left(\frac{P^2G(m_1+m_2)}{4\pi^2}\right)^{1/3} = 1.144 \times 10^{10}\,\mathrm{cm}, \tag{32}$$

and one can also compute $\dot{r} = 8.14 \times 10^{-5}$ cm/s.

(There are two other possibilities here that students might try. In some cases, one could also, less accurately, say that $r = Pv/2\pi$, where v is the value quoted in the text. This does not take account of the different masses of the stars though, and gives $r = 7.5 \times 10^9$ cm. To take into account different masses and thus a center of mass not at the center of the circle, one could solve for the semi-major axis, which gives $r = Pv/2\pi(m_1+m_2)/m2 = 1.142 \times 10^{10}\,\mathrm{cm}$. This is much closer to the right value.)

(e) *The correct orbital radius is 1.14×10^{10} cm. Using this and the paper's mass values gives $\dot{P} = 8.167 \times 10^{-12}$ s per s, very close to the value quoted in the discussion of the paper.*

(If one used the wrong relation for the radius, the resulting expressions all have a constant v, which is incorrect. The values one would obtain for the two scenarios above are $\dot{P} = 2.9 \times 10^{-11}$ s per s, and $\dot{P} = 5.48 \times 10^{-12}$ s per s, respectively.)

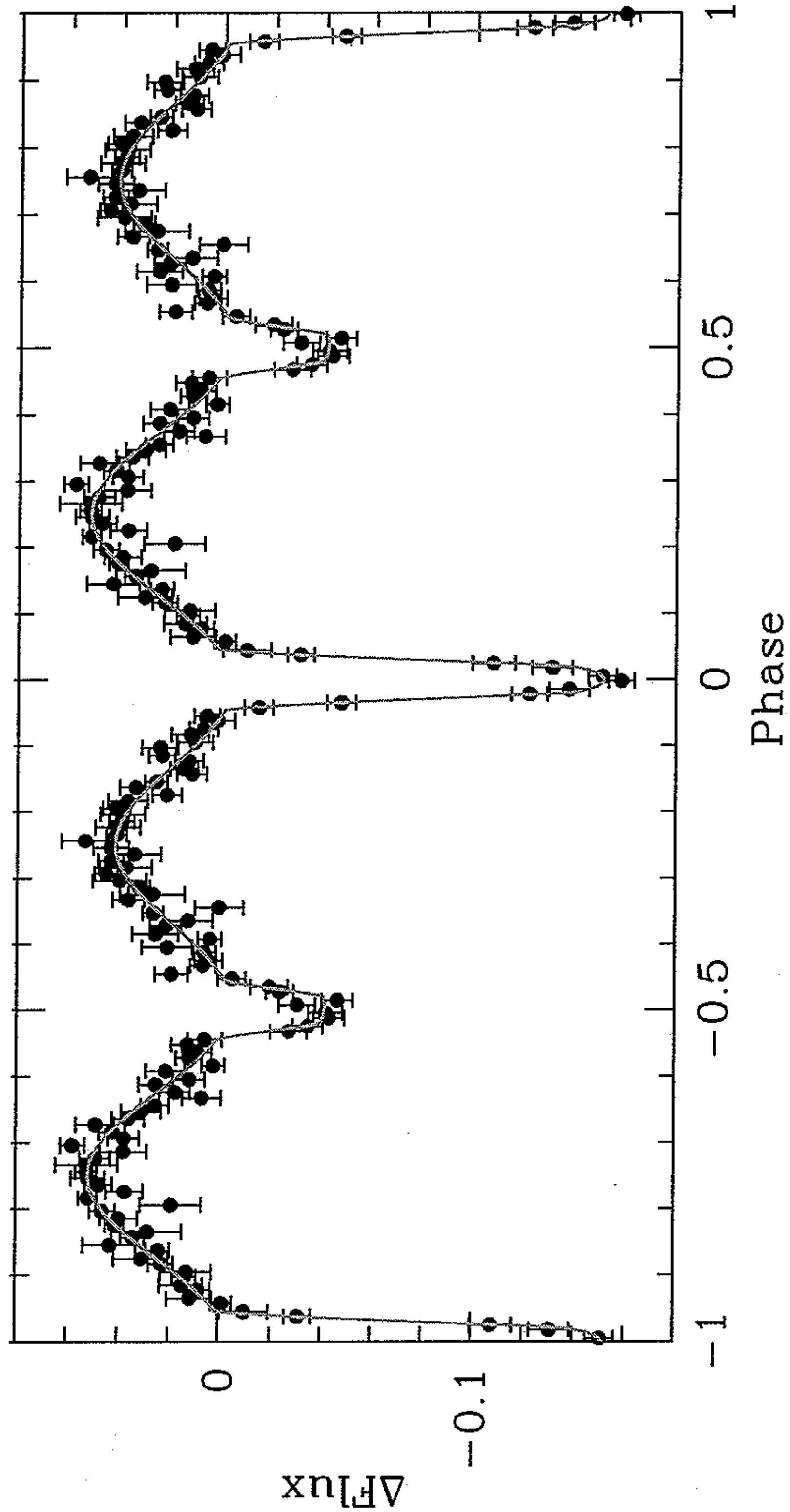

Figure 3: Folded photometric flux data versus orbital phase. The line is the best-fit model. These are data from the discovery paper of this system, Brown et al. 2011.