Cume # 351
Jason Jackiewicz
March 13, 2010

This exam deals with estimations of important stellar/solar quantities using concepts you know well. We are going to first try to derive a very precise value for the mass of the Sun, but then realize, believe it or not, that the limiting factor in obtaining a precise value is the inaccuracy with which we know the gravitational constant G. So after that disappointment, the rest of the exam is more or less back-of-the-envelope (with calculator) calculations of properties of stars like the Sun.

The *expected* passing grade is 70%, or about **44** out of a total of **64 points**. In Section II, you may choose between doing question 2 or 3, or do both. They are both 8 point questions, and you will be graded only on the one in which you do best. Note that question 3 also has 4 extra credit points to earn, so be aware of its potential of increasing your score.

Show all work clearly and legibly, and if you can't solve something completely, at least give an idea of how you might go about it. Don't use your calculators for any formulae or constants, only to calculate. Formulae and constants you may need are on the last page. Start each numbered problem on a new piece of paper. Take your time, think clearly, *read each sentence carefully*, ask for clarification, and best of luck to you!

I How to measure the solar mass $M_\odot$ from $GM_\odot$.

We are always told that the single most important property of a star is its mass. To derive this quantity for the Sun, we must begin with an observable. We'll take the time required for the Sun to return to a given location in the sky relative to the background stars after apparently travelling once around the ecliptic, from the point of view of an Earth observer. This time is $P = 31,558,150$ s. **(For this first section, keep a lot of decimal places.)**

1. **[2 points]**. What is the scientific name of P typically called?

2. **[7 points]**. Let's now turn to the equations of motion for the Earth and Sun as the Earth orbits the Sun. Take a position vector from some arbitrary point for the Sun ($\boldsymbol{r}_\oplus$) and the Earth ($\boldsymbol{r}_\odot$) and the position vector of Earth relative to the Sun $\boldsymbol{r} = \boldsymbol{r}_\oplus - \boldsymbol{r}_\odot$. The associated unit vector is $\hat{\boldsymbol{r}}$. Make a sketch of this situation with objects and position vectors ($\boldsymbol{r}$, $\boldsymbol{r}_\oplus$, and $\boldsymbol{r}_\odot$) clearly labeled. Then write down Newton's 2nd Law for both the Earth and Sun using $\boldsymbol{r}_\oplus$ and $\boldsymbol{r}_\odot$ (ignoring any other solar system bodies, thus only considering $m_\oplus$ and $M_\odot$.).

3. **[2 points]**. Now combine your two equations to find an expression for the acceleration of the position vector $\boldsymbol{r}$. If you knew how to solve your resulting equation, what do you expect the solution might be?

4. **[3 points]**. Write down Newton's version of Kepler's 3rd Law so that the left hand side only has the quantities G, $M_\odot$ and D^3, where D is the semimajor axis of an ellipse (of course, this says nothing about the right hand side). If you don't know the formula for Kepler's 3rd Law, get it from unit analysis!

5. **[2 points]**. We know that $m_\oplus$ is quite small compared to $M_\odot$, so we could get a fairly good approximation to the left hand side of the equation you just derived (as Kepler did) by ignoring small things. But, whatever approximation we take, we cannot get $M_\odot$ without D. The IAU defines $D \equiv 1\,\mathrm{AU} = 149\,597\,870.691\,\mathrm{km}$. Use this value and treat Earth to be much less massive than the Sun to find the first approximation of $GM_\odot$ **to 7 significant digits in cgs units!**

6. **[6 points]**. To get a more precise estimate of this quantity $GM_\odot$, we need to explicitly evaluate the ratio $m_\oplus/M_\odot$. To do this, it's necessary to compare the orbit of 2 objects, one in orbit around the Sun (let's take this object to be the Earth) with period P_{S} and semimajor axis D_{S}, and one in orbit around the Earth (let's take a small satellite) with period P_{E} and semimajor axis D_{E}. Derive an equation for $M_\odot/m_\oplus$ as a function of only P_{S}, D_{S}, P_{E}, and D_{E}.

7. **[4 points]**. We are getting there but we need orbital information for some satellite. For fun, let's take the recently launched Solar Dynamics Observatory (SDO - whose launch Michael saw). Currently, SDO has an apogee height (from Earth's surface) of 35533 km and perigee height of 20128 km. It orbits the Earth 1.37211406 times per day (it will eventually be geosynchronous). Using this information, show that $m_\oplus/M_\odot = 1/332\,960$.

8. **[2 points]**. Interestingly, if we used many satellites to reduce errors, the established value we would reach for $m_\oplus/M_\odot$ would converge to $1/332\,946$. Use this value to find the product of the gravitational constant and mass of the Sun (go back to your expression in question 4). At which decimal place does your answer differ from before.

9. **[2 points]**. Currently, the value of $GM_\odot$ is known to about 12 significant numbers, but the solar mass is not that well known because G is one of the most poorly known constants: only to 1 part in 10^4. Take $G = 6.67428(\pm 0.00067) \times 10^{-8}\,\mathrm{dyne\,cm^2\,g^{-2}}$ and (finally) compute the solar mass.

II Estimates of stellar quantities

Now we'll use the mass you just computed and the gravitational constant for the rest of the exam in some simple problems. For kicks, you can now just round off the mass you worked so hard to obtain to 3 significant digits. If you got it wrong, then use your wrong value correctly from here on. Remember, you need only do either #2 or #3, or both, and you will be graded on the best one.

1. **[21 points]**. Thermal timescale. Assume a star's only energy source is its gravitational potential energy.

   (a) Write down the equation for the gravitational force ($\mathrm{d}\boldsymbol{F}_{\mathrm{G}}$) of some infinitesimal mass shell $\mathrm{d}m$ located at radius r outside of interior mass M_r. Then write down the corresponding expression for the gravitational potential energy of the mass shell, i.e., the work done to bring the shell from infinity to position r.

   (b) Express the total gravitational potential energy as an integral over the radial displacement of the star of radius R, assuming many shells of thickness $\mathrm{d}r$ and that the mass is *uniformly distributed* throughout each of the shells.

(c) What do we need to know to solve this equation exactly?

(d) Make a simplifying approximation (with justification) so that you can solve the integral to find the total gravitational potential energy. Express your final answer in terms of G, M, and R.

(e) Therefore, what is the total *thermal* (kinetic) energy? (*Hint: Virial*).

(f) Compute an expression for the timescale for the release of this energy from the equation you just derived and any other quantity necessary. Express your answer in terms of scaled solar quantities, i.e., have only terms like $(f/f_\odot)^n$ for some property f.

(g) What is the commonly referred to name of this timescale of energy release? What is wrong with this hypothesis?

2. [**8 points**]. Nuclear timescale $\tau_{\rm nuc}$. Let's instead assume that nuclear fusion is the sole source of energy for a star such as the Sun. After computing the total available energy, find a timescale for the depletion of hydrogen for the Sun. We can assume that 100% of the mass of the Sun is hydrogen, but that only some approximate amount of it is involved in the fusion into helium. Approximate the reaction as 4H $\rightarrow$ He. Is this timescale reasonable? (Show all computations for energy conversion, do not just recall a value that you know without justification.)

3. [**8 points**]. Dynamical timescale $\tau_{\rm dyn}$.

   (a) Estimate the time it would take a particle to fall (from rest) a distance ℓ from the surface to the center of a star due to gravity. Ignore pressure. Express your answer again in terms of scaled solar values. What physical significance do you think this time represents (2 pts. extra credit)? Finally, since you have a scaled relationship into which is easy to plug generic stellar values, estimate the shortest and longest expected dynamical timescales based on what you know about the range of needed stellar parameters (min and max).

   (b) [***Extra credit:* 2 points**]. You may be wondering if this dynamical timescale has any connection to reality. Well, consider a small object orbiting a much larger one of mass M with semimajor axis R. Find a simple expression relating the period of this orbit to the dynamical timescale (the unscaled expression you just derived).

4. [**5 points**]. Say you want to estimate the pressure at the center of the Sun, *assuming hydrostatic equilibrium*. It's quite hard to guess a priori; is it about 50 dyne cm^{-2} or 5 billion dyne cm^{-2}? Take the *simplest* possible order-of-magnitude way of doing this, and come up with a number. Compare to the supposed real value of 2.4×10^{17} dyne cm^{-2}.

Cheat sheet

- Formulae:

$$\boldsymbol{F}_{\mathrm{G}} = G\frac{M_1 M_2}{r^2}\hat{\boldsymbol{r}} \tag{1}$$

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM\rho}{r^2} \tag{2}$$

- Constants:

$$c = 3\times 10^{10}\,\mathrm{cm\,s^{-1}} \tag{3}$$

$$R_\odot = 6.96\times 10^{10}\,\mathrm{cm} \tag{4}$$

$$R_\oplus = 6378.137\,\mathrm{km} \tag{5}$$

$$L_\odot = 3.9\times 10^{33}\,\mathrm{erg/s} \tag{6}$$

$$m_{\mathrm{proton}} = 1.00727646688\,\mathrm{amu} \tag{7}$$

$$m_{\mathrm{neutron}} = 1.00866491578\,\mathrm{amu} \tag{8}$$

$$m_{\mathrm{electron}} = 0.00054857991\,\mathrm{amu} \tag{9}$$

$$m_{\mathrm{He}} = 4.002603\,\mathrm{amu} \tag{10}$$

Cume # 351 – Solutions
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This exam deals with estimations of important stellar/solar quantities using concepts you know well. We are going to first try to derive a very precise value for the mass of the Sun, but then realize, believe it or not, that the limiting factor in obtaining a precise value is the inaccuracy with which we know the gravitational constant G. So after that disappointment, the rest of the exam is more or less back-of-the-envelope (with calculator) calculations of properties of stars like the Sun.

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I How to measure the solar mass $M_\odot$ from $GM_\odot$.

We are always told that the single most important property of a star is its mass. To derive this quantity for the Sun, we must begin with an observable. We'll take the time required for the Sun to return to a given location in the sky relative to the background stars after apparently travelling once around the ecliptic, from the point of view of an Earth observer. This time is $P = 31,558,150$ s. **(For this first section, keep a lot of decimal places.)**

1. **[2 points]. What is the scientific name of P typically called?**

   ANSWER: Translating to days, we see that $P = 365.25636$ days, so this is a sidereal year.

2. **[7 points]. Let's now turn to the equations of motion for the Earth and Sun as the Earth orbits the Sun. Take a position vector from some arbitrary point for the Sun ($\boldsymbol{r}_\oplus$) and the Earth ($\boldsymbol{r}_\odot$) and the position vector of Earth relative to the Sun $\boldsymbol{r} = \boldsymbol{r}_\oplus - \boldsymbol{r}_\odot$. The associated unit vector is $\hat{\boldsymbol{r}}$. Make a sketch of this situation with objects and position vectors ($\boldsymbol{r}$, $\boldsymbol{r}_\oplus$, and $\boldsymbol{r}_\odot$) clearly labeled. Then write down Newton's 2nd Law for both the Earth and Sun using $\boldsymbol{r}_\oplus$ and $\boldsymbol{r}_\odot$ (ignoring any other solar system bodies, thus only considering $m_\oplus$ and $M_\odot$.).**

   ANSWER: See Figure 1 for the geometry. The only force acting on Earth is the force due to gravity of the Sun. Thus

$$m_\oplus \frac{\mathrm{d}^2 \boldsymbol{r}_\oplus}{\mathrm{d}t^2} = -G\frac{m_\oplus M_\odot}{r^2}\hat{\boldsymbol{r}} \tag{1}$$

   directed towards the Sun (thereby negative). For the Sun, its acceleration due to Earth's gravity is expressed as

$$M_\odot \frac{\mathrm{d}^2 \boldsymbol{r}_\odot}{\mathrm{d}t^2} = +G\frac{m_\oplus M_\odot}{r^2}\hat{\boldsymbol{r}} \tag{2}$$

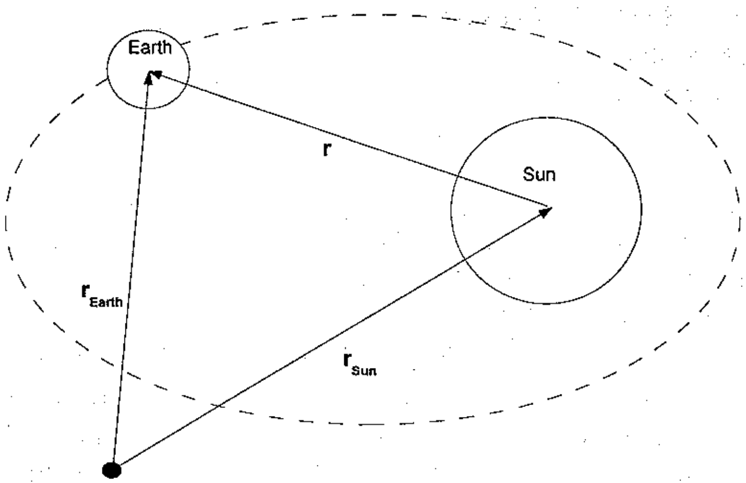

Figure 1: The geometry for the position vectors of the Earth and Sun for question I.2.

towards Earth.

3. **[2 points]. Now combine your two equations to find an expression for the acceleration of the position vector r. If you knew how to solve your resulting equation, what do you expect the solution might be?**

   ANSWER: This is simply

$$\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = \frac{\mathrm{d}^2\boldsymbol{r}_\oplus}{\mathrm{d}t^2} - \frac{\mathrm{d}^2\boldsymbol{r}_\odot}{\mathrm{d}t^2} = -G\frac{(M_\odot + m_\oplus)}{r^2}\hat{\boldsymbol{r}}. \tag{3}$$

   This equation's solution would be (better be!) an ellipse with the center of mass at one focus.

4. **[3 points]. Write down Newton's version of Kepler's 3rd Law so that the left hand side only has the quantities G, $M_\odot$ and D^3, where D is the semimajor axis of an ellipse (of course, this says nothing about the right hand side). If you don't know the formula for Kepler's 3rd Law, get it from unit analysis!**

   ANSWER:

$$P^2 = \frac{4\pi^2 D^3}{G(M_\odot + m_\oplus)}, \tag{4}$$

$$P^2 = \frac{4\pi^2 D^3}{GM_\odot(1 + m_\oplus/M_\odot)}, \tag{5}$$

$$\frac{GM_\odot}{D^3} = \frac{4\pi^2}{P^2(1 + m_\oplus/M_\odot)}. \tag{6}$$

5. **[2 points]. We know that $m_\oplus$ is quite small compared to $M_\odot$, so we could get a fairly good approximation to the left hand side of the equation you just derived (as Kepler did) by ignoring small things. But, whatever approximation we take, we cannot get $M_\odot$ without D. The IAU defines $D \equiv 1\,\mathrm{AU} = 149\,597\,870.691\,\mathrm{km}$. Use this value and treat Earth to be much less massive than the Sun to find the first approximation of $GM_\odot$ to 7 significant digits in cgs units!**

   ANSWER:

$$GM_\odot \approx \frac{4\pi^2 D^3}{P^2} = 1.327128 \times 10^{26}\,\mathrm{cm}^3\,\mathrm{s}^{-2} \tag{7}$$

6. [6 points]. To get a more precise estimate of this quantity $GM_\odot$, we need to explicitly evaluate the ratio $m_\oplus/M_\odot$. To do this, it's necessary to compare the orbit of 2 objects, one in orbit around the Sun (let's take this object to be the Earth) with period $P_{\rm S}$ and semimajor axis $D_{\rm S}$, and one in orbit around the Earth (let's take a small satellite) with period $P_{\rm E}$ and semimajor axis $D_{\rm E}$. Derive an equation for $M_\odot/m_\oplus$ as a function of only $P_{\rm S}$, $D_{\rm S}$, $P_{\rm E}$, and $D_{\rm E}$.

ANSWER: For the Earth orbiting the Sun, we have

$$P_{\rm S}^2 \sim \frac{D_{\rm S}^3}{M_\odot + m_\oplus}, \tag{8}$$

and for the satellite orbiting Earth

$$P_{\rm E}^2 \sim \frac{D_{\rm E}^3}{m_\oplus}. \tag{9}$$

The satellite mass is perfectly ignorable. Dividing equation (9) by (8) gives

$$\left(\frac{P_{\rm E}}{P_{\rm S}}\right)^2 = \left(\frac{D_{\rm E}}{D_{\rm S}}\right)^3 \cdot \left(1 + \frac{M_\odot}{m_\oplus}\right), \tag{10}$$

$$\frac{M_\odot}{m_\oplus} = \left(\frac{P_{\rm E}}{P_{\rm S}}\right)^2 \left(\frac{D_{\rm S}}{D_{\rm E}}\right)^3 - 1 \tag{11}$$

7. [4 points]. We are getting there but we need orbital information for some satellite. For fun, let's take the recently launched Solar Dynamics Observatory (SDO - whose launch Michael saw). Currently, SDO has an apogee height (from Earth's surface) of 35533 km and perigee height of 20128 km. It orbits the Earth 1.37211406 times per day (it will eventually be geosynchronous). Using this information, show that $m_\oplus/M_\odot = 1/332\,960$.

ANSWER: We need to solve Equation (11). For SDO, the period

$$P_{\rm E} = \frac{24 \cdot 60 \cdot 60}{1.37211406} = 62968.5261\,{\rm s}. \tag{12}$$

The semimajor axis will be the average of perigee and apogee plus the radius of Earth (from appendix)

$$D_{\rm E} = \frac{35533 + 20126}{2} + 6378.137 = 34208.637. \tag{13}$$

Plugging these into Equation (11) gives $M_\odot/m_\oplus = 332\,960$. If one "forgot" to use the Earth's radius, the result is $618\,421$.

8. [2 points]. Interestingly, if we used many satellites to reduce errors, the established value we would reach for $m_\oplus/M_\odot$ would converge to $1/332\,946$. Use this value to find the product of the gravitational constant and mass of the Sun (go back to your expression in question 4). At which decimal place does your answer differ from before.

ANSWER: Using this value of the ratio of masses

$$GM_\odot = \frac{4\pi^2 D^3}{P^2(1 + m_\oplus/M_\odot)} = 1.327124 \times 10^{26}\,{\rm cm^3\,s^{-2}}. \tag{14}$$

We see a difference in the sixth decimal place. Not great, but ...

9. [2 points]. **Currently, the value of $GM_\odot$ is known to about 12 significant numbers, but the solar mass is not that well known because G is one of the most poorly known constants: only to 1 part in 10^4. Take $G = 6.67428(\pm 0.00067) \times 10^{-8}\,\mathrm{dyne\,cm^2\,g^{-2}}$ and (finally) compute the solar mass.**

   ANSWER:

$$M_\odot = \frac{1.327124 \times 10^{26}}{6.67428 \times 10^{-8}} = 1.9884 \times 10^{33}\,\mathrm{g}, \tag{15}$$

   which we can only keep to the fourth decimal place due to the errors.

II Estimates of stellar quantities

Now we'll use the mass you just computed and the gravitational constant for the rest of the exam in some simple problems. For kicks, you can now just round off the mass you worked so hard to obtain to 3 significant digits. If you got it wrong, then use your wrong value correctly from here on. Remember, you need only do either #2 or #3, or both, and you will be graded on the best one.

1. [21 points]. **Thermal timescale. Assume a star's only energy source is its gravitational potential energy.**

   (a) **Write down the equation for the gravitational force ($\mathrm{d}F_\mathrm{G}$) of some infinitesimal mass shell $\mathrm{d}m$ located at radius r outside of interior mass M_r. Then write down the corresponding expression for the gravitational potential energy of the mass shell, i.e., the work done to bring the shell from infinity to position r.**

   ANSWER: (NOTE: See A. Klypin solution page at the end for answers to parts a-d with an alternative approach - both final answers agree). The gravitational force on the shell is

$$\mathrm{d}\boldsymbol{F}_\mathrm{G} = \boldsymbol{F}_\mathrm{G}^\mathrm{shell} = -G\frac{M_r \mathrm{d}m}{r^2}\hat{\boldsymbol{r}}, \tag{16}$$

   where $\hat{\boldsymbol{r}}$ is outward. The total potential energy for the shell is $U_\mathrm{G}^\mathrm{shell} = -\int_\infty^r \boldsymbol{F}_\mathrm{G}^\mathrm{shell} \cdot \mathrm{d}\boldsymbol{r}'$, or in this case,

$$\mathrm{d}U_\mathrm{G} = U_\mathrm{G}^\mathrm{shell} = GM_r\mathrm{d}m \int_\infty^r \frac{1}{r'^2}\,\mathrm{d}r' = -G\frac{M_r\mathrm{d}m}{r}, \tag{17}$$

   where, again, all the mass is interior to the point r, thus outside of the integration regime.

   (b) **Express the total gravitational potential energy as an integral over the radial displacement of the star of radius R, assuming many shells of thickness $\mathrm{d}r$ and that the mass is *uniformly distributed* throughout each of the shells.**

   ANSWER: Since the mass is said to be uniformly distributed, we can assume a constant density within a shell, substitute the mass $\mathrm{d}m = \rho\,\mathrm{d}V = 4\pi r^2\rho\mathrm{d}r$, and integrate over all shells to get

$$U_\mathrm{G}^\mathrm{tot} = \int_0^R U_\mathrm{G}^\mathrm{all\,shells} = -4\pi G \int_0^R M_r\rho(r)\,r\,\mathrm{d}r. \tag{18}$$

(c) **What do we need to know to solve this equation exactly?**

ANSWER: We would need to know the radial dependence of the mass or density throughout the star.

(d) **Make a simplifying approximation (with justification) so that you can solve the integral to find the total gravitational potential energy. Express your final answer in terms of G, M, and R.**

ANSWER: One thing to do would be to assume that the density is constant, such that the total mass can be expressed by the average value of the density of the star $\overline{\rho}$. Then $M_r \approx (4/3)\pi\overline{\rho}r^3$ and

$$U_{\mathrm{G}} = -\frac{16\pi^2}{3}G\overline{\rho}^2 \int_0^R r^4 \,\mathrm{d}r = -\frac{16\pi^2}{15}G\overline{\rho}^2 R^5. \tag{19}$$

Let the density $\overline{\rho} = 3/4(M/R^3)$ to finally get

$$U_{\mathrm{G}} = -\frac{3}{5}\frac{GM^2}{R}. \tag{20}$$

(e) **Therefore, what is the total *thermal* (kinetic) energy? (*Hint: Virial*).**

ANSWER: The Virial theorem says that $2E_{\mathrm{T}} + U_{\mathrm{G}} = 0$, therefore

$$E_{\mathrm{T}} = \frac{3}{10}\frac{GM^2}{R}. \tag{21}$$

(f) **Compute an expression for the timescale for the release of this energy from the equation you just derived and any other quantity necessary. Express your answer in terms of scaled solar quantities, i.e., have only terms like $(f/f_\odot)^n$ for some property f.**

ANSWER: A time scale τ for all the energy radiated away can be obtained by dividing the total energy by the luminosity:

$$\tau = \frac{E_{\mathrm{T}}}{L} = \frac{3}{10}\frac{GM^2}{RL}. \tag{22}$$

In terms of solar values we get

$$\begin{aligned} \tau &= \frac{3}{10}\frac{GM_\odot^2}{R_\odot L_\odot} \cdot \left(\frac{M}{M_\odot}\right)^2 \left(\frac{R}{R_\odot}\right)^{-1} \left(\frac{L}{L_\odot}\right)^{-1} && (23) \\ &\approx 10\,\text{million yr} \cdot \left(\frac{M}{M_\odot}\right)^2 \left(\frac{R}{R_\odot}\right)^{-1} \left(\frac{L}{L_\odot}\right)^{-1} && (24) \end{aligned}$$

(g) **What is the commonly referred to name of this timescale of energy release? What is wrong with this hypothesis?**

ANSWER: Kelvin-Helmholtz timescale. Nuclear fusion is the real source.

2. [8 points]. **Nuclear timescale $\tau_{\rm nuc}$. Let's instead assume that nuclear fusion is the sole source of energy for a star such as the Sun. After computing the total available energy, find a timescale for the depletion of hydrogen for the Sun. We can assume that 100% of the mass of the Sun is hydrogen, but that only some approximate amount of it is involved in the fusion into helium. Approximate the reaction as 4H $\rightarrow$ He. Is this timescale reasonable? (Show all computations for energy conversion, do not just recall a value that you know without justification.)**

ANSWER: Assume that only 10% of the hydrogen mass is used. Since 4 hydrogen atoms have more mass than a helium atom, some goes into energy. This amount is $(m_{4\rm H} - m_{\rm He})/m_{4\rm H} \approx$ 0.7%, *so we have a* 0.7% *efficiency. Then*

$$
\begin{aligned}
E_{\rm nuc} &\approx \text{factor} \cdot M_\odot c^2 && (25)\\
&= 0.007 \cdot 0.1 \cdot (1.99 \times 10^{33}\,\text{g})(3 \times 10\,\text{cm}\,\text{s}^{-1})^2 && (26)\\
&= 1.25 \times 10^{51}\,\text{erg}. && (27)
\end{aligned}
$$

Then, as before

$$\tau_{\rm nuc} = E_{\rm nuc}/L_\odot = 3.2 \times 10^{17}\,\text{s} \approx 1.0 \times 10^{10}\,\text{year}. \tag{28}$$

So about 10 billion years is much more realistic.

3. [8 points]. **Dynamical timescale $\tau_{\rm dyn}$.**

   (a) **Estimate the time it would take a particle to fall (from rest) a distance ℓ from the surface to the center of a star due to gravity. Ignore pressure. Express your answer again in terms of scaled solar values. What physical significance do you think this time represents (2 pts. extra credit)? Finally, since you have a scaled relationship into which is easy to plug generic stellar values, estimate the shortest and longest expected dynamical timescales based on what you know about the range of needed stellar parameters (min and max).**

   ANSWER: We know that $\ell = 1/2gt^2$, $\ell = R$, *and that* $g = GM/R^2$. *So*

$$
\begin{aligned}
\tau_{\rm dyn} &= \left(\frac{2\ell}{g}\right)^{1/2} = \left(\frac{2R^3}{GM}\right)^{1/2} = \left(\frac{2R_\odot^3}{GM_\odot}\right)^{1/2} \cdot \left(\frac{R}{R_\odot}\right)^{3/2} \left(\frac{M}{M_\odot}\right)^{-1/2} && (29)\\
&\approx 38\,\text{min} \cdot \left(\frac{R}{R_\odot}\right)^{3/2} \left(\frac{M}{M_\odot}\right)^{-1/2}. && (30)
\end{aligned}
$$

   The dynamical timescale represents (approximately) the fundamental mode oscillation period of a star, since the star needs this amount of time to react to any global perturbations that may travel throughout the star.

   We know roughly that stars range from about $0.01R_\odot$ *to* $1000R_\odot$, *and from about* $0.1M_\odot$ *to* $100M_\odot$. *The shortest timescale would be for the smallest radius and largest mass, giving* $\tau_{\rm dyn} \approx 0.1$ *s. The longest timescale for the largest but least massive is* $\tau_{\rm dyn} \approx 5$ *yr.*

   (b) [*Extra credit:* **2 points]. You may be wondering if this dynamical timescale has any connection to reality. Well, consider a small object orbiting a much larger one of mass M with semimajor axis R. Find a simple expression relating the period of this orbit to the dynamical timescale (the unscaled expression you just derived).**

   ANSWER: The period of this orbit is $P^2 = 4\pi^2 R^3/(GM)$. *We easily see from equation (29) that* $P = 2\pi\tau_{\rm dyn}$. *Thus, it does connect with a timescale for general motion in a gravitational field.*

4. [5 points]. **Say you want to estimate the pressure at the center of the Sun, *assuming hydrostatic equilibrium.* It's quite hard to guess a priori; is it about 50 dyne cm^{-2} or 5 billion dyne cm^{-2}? Take the *simplest* possible order-of-magnitude way of doing this, and come up with a number. Compare to the supposed real value of 2.4×10^{17} dyne cm^{-2}.**

ANSWER: The absolute easiest way to do this is to take the hydrostatic equilibrium equation and replace $\mathrm{d}P/\mathrm{d}r$ *by* $-P_c/R_\odot$*, replace* M *by* $M_\odot$*, replace* ρ *by its average value* $\rho \approx M_\odot/R_\odot^3$ *and* r *by* $R_\odot$*. We are then left with*

$$\frac{P_c}{R_\odot} \approx \frac{GM_\odot^2}{R_\odot^5}, \tag{31}$$

or

$$P_c \approx \frac{GM_\odot^2}{R_\odot^4} \approx 1.13 \times 10^{16}\,\mathrm{dyne\,cm^{-2}}. \tag{32}$$

It's a little more than an order of magnitude off the real value, but not so bad considering how easily we solved a difficult equation. Note: if they kept the $4/3\pi$ *in the density equation, the final answer would be* $P_c = 2.69 \times 10^{15}$ *dyne cm*$^{-2}$.

Cheat sheet

- Formulae:

$$\boldsymbol{F}_{\mathrm{G}} = G\frac{M_1 M_2}{r^2}\hat{r} \tag{33}$$

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM\rho}{r^2} \tag{34}$$

- Constants:

$$c = 3 \times 10^{10}\,\mathrm{cm\,s^{-1}} \tag{35}$$

$$R_{\odot} = 6.96 \times 10^{10}\,\mathrm{cm} \tag{36}$$

$$R_{\oplus} = 6378.137\,\mathrm{km} \tag{37}$$

$$L_{\odot} = 3.9 \times 10^{33}\,\mathrm{erg/s} \tag{38}$$

$$m_{\mathrm{proton}} = 1.00727646688\,\mathrm{amu} \tag{39}$$

$$m_{\mathrm{neutron}} = 1.00866491578\,\mathrm{amu} \tag{40}$$

$$m_{\mathrm{electron}} = 0.00054857991\,\mathrm{amu} \tag{41}$$

$$m_{\mathrm{He}} = 4.002603\,\mathrm{amu} \tag{42}$$