Cume # 343
Jason Jackiewicz
April 4, 2009

This exam deals with something we always love to conveniently ignore — **magnetic fields!** There is no accompanying article to peruse. You will work through problems and hopefully learn as you proceed. Most of the questions do not expect you to have much prior knowledge of this subject, except some basic E&M and solar/stellar parameters. You will derive things mainly from first principles, leading up to an application in the final section. Thus the exam is, as its name suggests, cumulative. There is an appendix towards the end with any formulas or constants you may need.

The *expected* passing grade is about 65%, or **55** out of a total of **85 points**. There are a couple questions that are for extra credit and 5 possible points. These are not counted as part of the 85 points. *It is advisable that you attempt these last, if you have time left. Partial credit applies to these too.* Most of the problems are of the "show this" type, so that if you can't derive the result, at least you'll have the answer in hand to complete subsequent parts. Show all work clearly and legibly, and if you can't solve something completely, at least give an idea of how you might go about it. For questions where you must provide a name, and can't, at least give what units it should have or anything helpful like that. Start each numbered problem on a new piece of paper. Take your time, think clearly, ask for clarification, and best of luck to you!

I Magnetic-field configurations

For many planetary and stellar situations, the magnetic field outlines the structure of various phenomena and it is therefore important to know how to visualize *magnetic field lines* (lines of force). For any known magnetic field $\boldsymbol{B} = (B_x, B_y, B_z)$, the magnetic field lines are given by

$$\frac{\mathrm{d}x}{B_x} = \frac{\mathrm{d}y}{B_y} = \frac{\mathrm{d}z}{B_z}. \tag{1}$$

Boldface quantities will denote vectors throughout. Equation (1) is a system of differential equations defining curves in three-dimensional space, whose solution is found by **integration**. These curves are called field lines and are tangent to the direction of the field at each point. The spacing of the lines corresponds to the strength of the field. The closer, the stronger. The lines are also required to have arrows to denote the direction of the field.

1. [**5 points**]. As an example, take a look at Figure 1. This plot shows the field lines for the case of $\boldsymbol{B} = (y, x, 0)$ as well as the normalized magnitude of the field $B = |\boldsymbol{B}|$. Around the edges where the field lines are close together is where the field is relatively strong. Also note the directional arrows.

   (a) Show that this magnetic field is a physically plausible one (i.e., no sources or sinks – remember one of Maxwell's equations!).

   (b) By using equation (1), derive the correct expression for the field lines for this example and then use it to explain why Figure 1 looks the way it does. Also, explain why the arrows do point in the directions given.

2. **[10 points]**. Now try one for yourself. Take the field $\boldsymbol{B} = (0, x^2, 0)$.

   (a) Check first that this is a physical magnetic field.

   (b) Solve for the equation of the field lines.

   (c) Make two plots. The first one is a 1D plot of the magnitude of the field as a function of position. The second one should be a sketch of the field lines in the $x - y$ plane. Make sure to consider the field as it gets stronger and don't forget to label the figure with arrows denoting the direction of the field lines.

II Derivation of magnetohydrodynamics (MHD)

Now we will derive some useful, physical equations that determine the time evolution of a magnetic field.

1. **[10 points]**. Consider two of Maxwell's equations (in cgs units):

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}, \tag{2}$$

$$\nabla \times \boldsymbol{B} = \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} + \frac{4\pi}{c}\boldsymbol{j}. \tag{3}$$

   The electric field is $\boldsymbol{E}$ and current density is $\boldsymbol{j}$. These equations are usually valid in astrophysical fluids or gases of plasma.

   (a) To whom are the names of each of these two equations attributed (after Maxwell's extension of them)? They are named after two very famous physicists.

   (b) Do a unit analysis of equation (2) using a characteristic length scale L and time scale t. Also assume a characteristic speed $u = L/t$ that is **non-relativistic**. What can you immediately qualitatively conclude about the magnitudes of B and E?

   (c) Do a similar analysis for equation (3). Under the same approximation that you discerned in (b), show that you can safely ignore one of the two terms on the RHS side of that equation.

   (d) **[*Extra credit:* 2 points]**. What is the common name of that 'neglected' term?

2. **[35 points]**. Now consider another relationship between the current density and the electromagnetic fields:

$$\boldsymbol{j} = \sigma\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}\right), \tag{4}$$

   where σ is the conductivity and $\boldsymbol{v}$ is the velocity of the plasma viewed from a rest frame.

   (a) What 'law' is equation (4) usually referred to as? Using equations (2), (3), (4) and any approximations you've made, derive the *induction equation* below solely in terms of the magnetic field:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}. \tag{5}$$

   (b) What did you find for the coefficient η? What units must it have? What physical interpretation does the first term on the RHS of equation (5) have? The second term?

(c) Consider a fluid at rest in equation (5). Define a time, τ_d, using η and any characteristic scales, as before. It can be shown (for a hydrogen plasma) that $\eta = 10^9\, T^{-3/2}\,\mathrm{m}^2\,\mathrm{s}^{-1}$, where T is the temperature, in Kelvin. What would the time τ_d be if we take the magnetic field in a sunspot (assume a plasma temperature a bit less than the surrounding photosphere and a ball-park size for a sunspot)? Comment on the value that you find in terms of the physics and time and length scales. What word do you think the "d" in τ_d might stand for?

(d) Astrophysical fluids and plasmas are not generally at rest, however. Therefore, the induction equation is really a struggle between the relative strengths of the two terms. Define an appropriate order-of-magnitude ratio of the two terms on the RHS in equation (5), called the *magnetic Reynolds number*, R_M, using any previous scales and quantities, and show that for typical solar values – try a sunspot again – it is enormous compared to 1 (if it's small, you've probably got your ratio upside-down, so re-define it). What does this imply for the conductivity of typical stars? How would the magnetic Reynolds number change in the corona of the Sun compared to the sunspot (show and describe this with simple hand-wavy arguments)?

(e) Since the Reynolds number is so large in most cases, we can safely neglect one of the terms in the induction equation. First write down the resulting induction equation, and explain the physical significance of it in terms of dynamics. (*Hint: For its significance, think about a plasma at rest.*)

(f) [***Extra credit*:** **3 points**]. Current 3D numerical simulations of convection in stars are only able to employ Reynolds numbers of roughly $\sim 10^3$ or so, much smaller than what we've seen to be the typical values. Based on your derivations, what factors may be limiting pushing this value higher in these simulations? Can you think of any numerical schemes or tricks to get around this obstacle to make the equations in the simulations more physical?

Congratulations! You've just (hopefully) derived a good deal of ideal magnetohydrodynamics.

III Applications

1. [**25 points**]. We've been talking about the magnetic field so much, let's find out if it's actually important for stars after all! The above MHD equations need to be coupled to an equation of motion to be useful. Let's define it as

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla}P + \boldsymbol{g} + \frac{1}{\rho c}\boldsymbol{j}\times\boldsymbol{B}, \tag{6}$$

where P is the gas pressure, ρ is mass density, and $\boldsymbol{g}$ is the acceleration due to gravity.

(a) What is the term $c^{-1}\boldsymbol{j}\times\boldsymbol{B}$ commonly called?

(b) Consider a stationary gas where the effects of gravity are not significant over the length scales involved compared to the other terms. Using equation (6) and any other ones you may need, define and derive what's called the plasma β parameter:

$$\beta = \frac{4\pi P}{B^2}, \tag{7}$$

where $B = |\boldsymbol{B}|$.

(c) The plasma β is an *approximate* measure of the relative strength of the gas pressure to the magnetic pressure. Let's find some values of it for the Sun. Consider the Sun to be pure, fully ionized hydrogen and an ideal gas. Show that the plasma β parameter is

$$\beta \approx 7 \times 10^{-15}\, n_{\mathrm{e}} T B^{-2}, \tag{8}$$

where n_{e} is the electron number density, and B is usually given in units of gauss (G).

(d) Use the above equation to describe, for a star like the Sun, when we can ignore magnetic fields or when magnetic fields dominate. Consult Figures 2 and 3 for some values you may need. Use your best guesses for all the quantities you do not have at hand. Consider specifically β in the (*i*) radiation zone, (*ii*) the 'quiet' photosphere, (*iii*) a sunspot, and (*iv*) the corona. Explain why, physically, these values of β make sense to you, as you work through each region.

Appendix: Useful Items

- For any vector $\boldsymbol{A}$:

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) - (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\boldsymbol{A} \tag{9}$$

- The ideal gas law is

$$P = \frac{\rho R T}{\mu}, \tag{10}$$

where $R = 8.3 \times 10^7\,\mathrm{cm^2 s^{-2} K^{-1}}$ is the gas constant, ρ is the mass density, and μ is related to the mean molecular weight $\overline{m}$ by

$$\overline{m} = \frac{n_\mathrm{p} m_\mathrm{p} + n_\mathrm{e} m_\mathrm{e}}{n_\mathrm{p} + n_\mathrm{e}} = \mu m_\mathrm{p}. \tag{11}$$

The ns are number particle densities, the ms are masses, and the subscripts p and e denote proton and electron, respectively.

- Mass of a proton $= 1.67 \times 10^{-24}$ g.

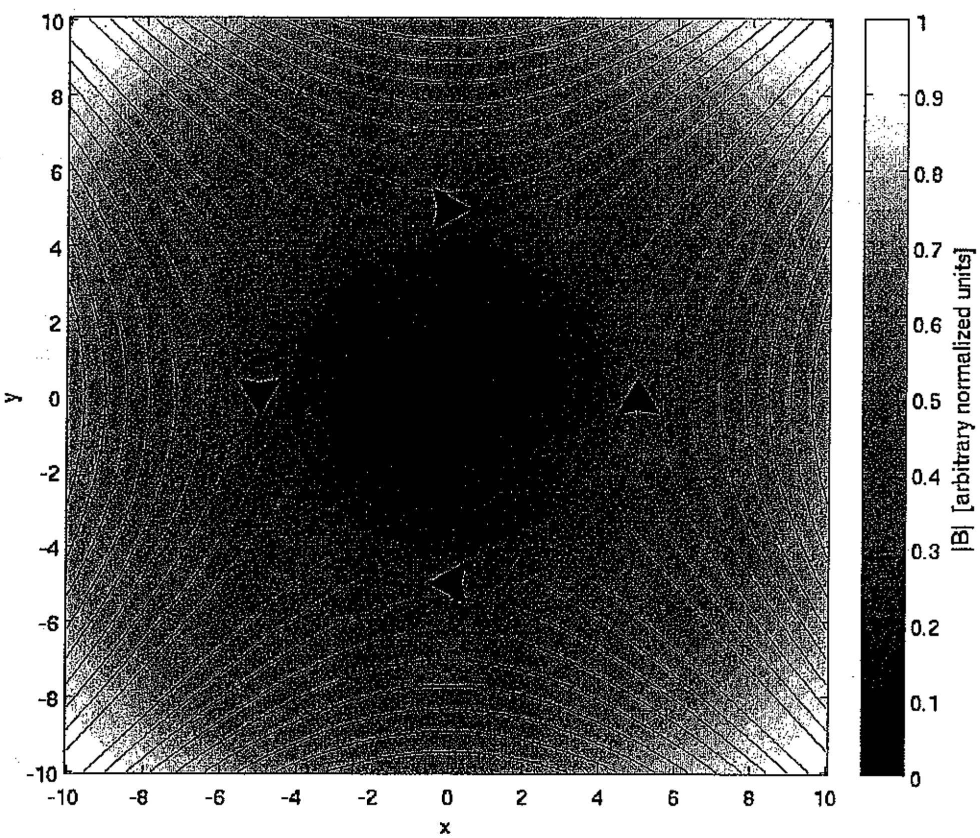

Figure 1: Field lines and directions for $\boldsymbol{B} = (y, x, 0)$. The background gray scale is the magnitude of the magnetic field, $|\boldsymbol{B}|$, normalized to unity. See problem I.1

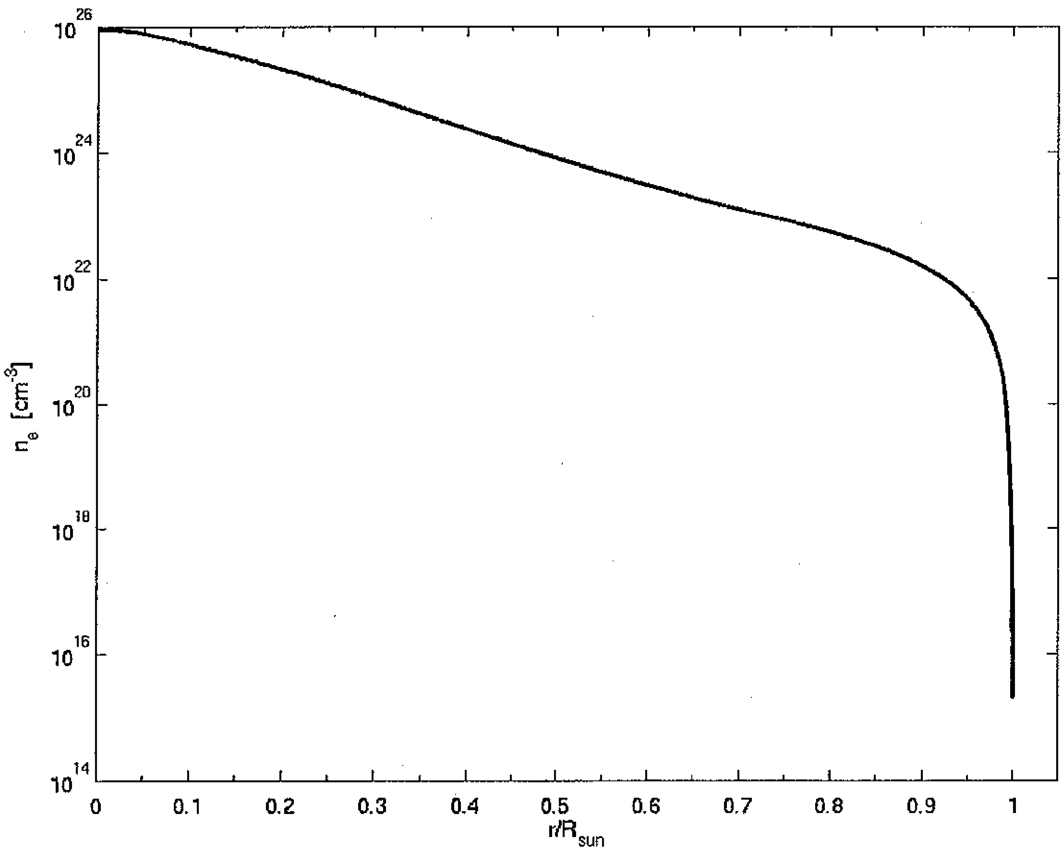

Figure 2: Electron number density versus fractional radius from a model of the Sun. See Problem III.1d.

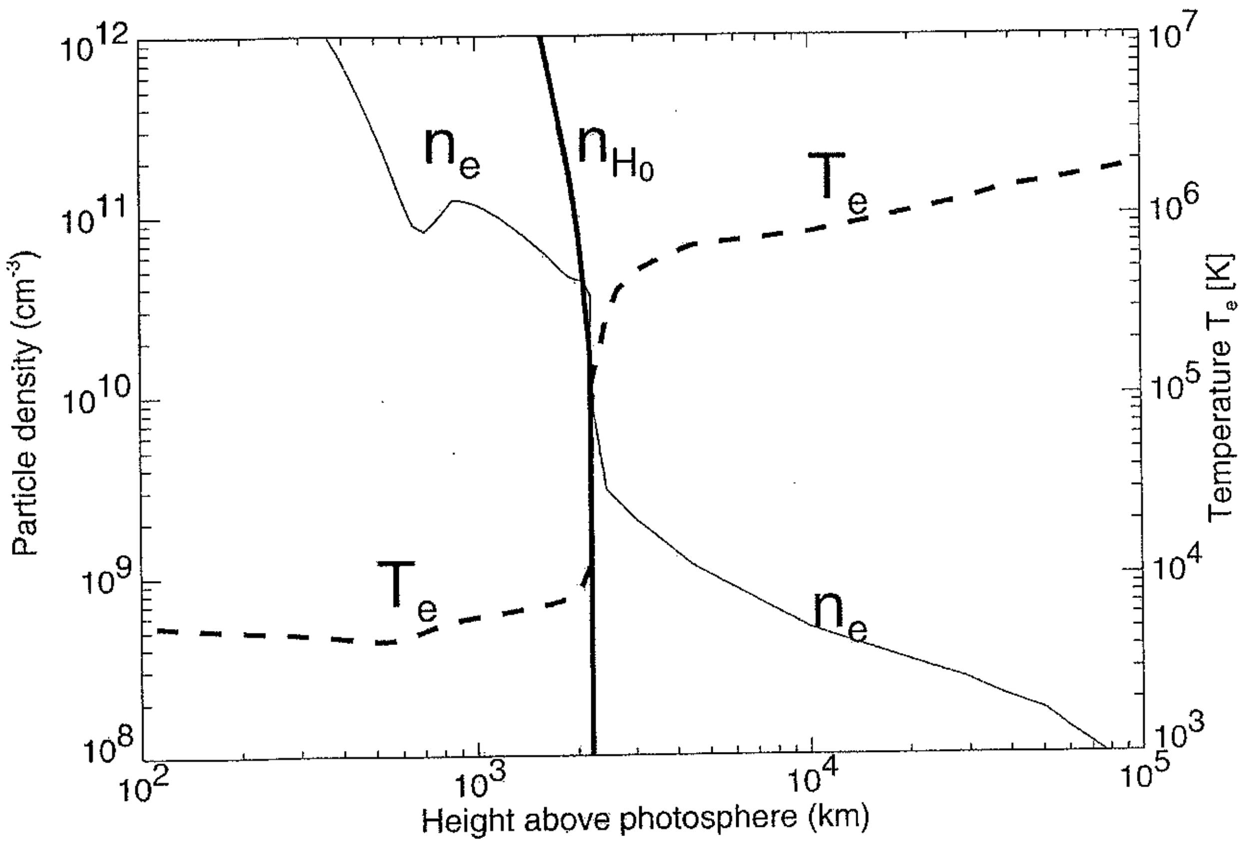

Figure 3: Electron number density and temperature versus height in the solar atmosphere. Ignore n_{H_0} and the thick vertical line. From Aschwanden 2006. See Problem III.1d.

Cume # 343 – Solutions
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I Magnetic-field configurations

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$$\frac{\mathrm{d}x}{B_x} = \frac{\mathrm{d}y}{B_y} = \frac{\mathrm{d}z}{B_z}. \tag{1}$$

Boldface quantities will denote vectors throughout. Equation (1) is a system of differential equations defining curves in three-dimensional space, whose solution is found by **integration.** These curves are called field lines and are tangent to the direction of the field at each point. The spacing of the lines corresponds to the strength of the field. The closer, the stronger. The lines are also required to have arrows to denote the direction of the field.

1. [**5 points**]. As an example, take a look at Figure 1. This plot shows the field lines for the case of $\boldsymbol{B} = (y, x, 0)$ as well as the normalized magnitude of the field $B = |\boldsymbol{B}|$. Around the edges where the field lines are close together is where the field is relatively strong. Also note the directional arrows.

   (a) Show that this magnetic field is a physically plausible one (i.e., no sources or sinks – remember one of Maxwell's equations!).

   ANSWER: No sources or sinks is the solenoidal constraint:

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 = \frac{\partial}{\partial x} y + \frac{\partial}{\partial y} x \equiv 0. \tag{2}$$

Another way some students tried was to use the integral form of the same equation:

$$\oint \boldsymbol{B} \cdot \mathrm{d}A = 0, \tag{3}$$

where $\mathrm{d}A$ **is some unit area vector. Ok ... if you wish. The way to do this then would be to switch to polar coordinates:**

$$\begin{aligned}
\oint \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{A} &= \oint B\,\mathrm{d}A\cos\theta && (4)\\
\mathrm{d}A &= \mathrm{d}x\mathrm{d}y = r\mathrm{d}r\mathrm{d}\theta && (5)\\
\int_0^r \int_0^{2\pi} r^2\cos\theta\mathrm{d}r\mathrm{d}\theta &\equiv 0, && (6)
\end{aligned}$$

where we used the fact that $B = \sqrt{y^2 + x^2} = r$**. It's the integral over** θ **that makes it all vanish.**

(b) By using equation (1), derive the correct expression for the field lines for this example and then use it to explain why Figure 1 looks the way it does. Also, explain why the arrows do point in the directions given.

ANSWER: Onlys consider the $x - y$ **plane. Set up the ratio:**

$$\begin{aligned}
\frac{\mathrm{d}x}{B_x} &= \frac{\mathrm{d}y}{B_y}, && (7)\\
\frac{\mathrm{d}x}{y} &= \frac{\mathrm{d}y}{x}, && (8)\\
\int x\,\mathrm{d}x &= \int y\,\mathrm{d}y, && (9)\\
x^2/2 &= y^2/2 + C, \qquad \text{or} && (10)\\
x^2 - y^2 &= C = \text{const} && (11)
\end{aligned}$$

This describes an equation of a hyperbola, which is evident from Figure 1. The lines are spaced far apart near the origin, and closer together as x **and** y **are increased in absolute magnitude. The arrows are straightforward. Along the line** $x = 0$**, for positive (negative)** y **the arrows should point in the positive + (negative -) direction. Along the** $y = 0$ **line, the situation is similar.**

2. **[10 points]**. Now try one for yourself. Take the field $\boldsymbol{B} = (0, x^2, 0)$.

(a) Check first that this is a physical magnetic field.

ANSWER:

$$\nabla \cdot \boldsymbol{B} = 0 = \frac{\partial}{\partial y}x^2 \equiv 0. \tag{12}$$

(b) Solve for the equation of the field lines.

ANSWER:

$$\frac{dx}{dy} = \frac{0}{x^2} \implies x = \text{const.} \tag{13}$$

This means the field lines will be along the y direction.

(c) Make two plots. The first one is a 1D plot of the magnitude of the field as a function of position. The second one should be a sketch of the field lines in the $x - y$ plane. Make sure to consider the field as it gets stronger and don't forget to label the figure with arrows denoting the direction of the field lines.

ANSWER: See Figures 2 and 3. The arrows will always point in the positive direction because B_y is never negative.

II Derivation of magnetohydrodynamics (MHD)

Now we will derive some useful, physical equations that determine the time evolution of a magnetic field.

1. [**10 points**]. Consider two of Maxwell's equations (in cgs units):

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}, \tag{14}$$

$$\nabla \times \boldsymbol{B} = \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} + \frac{4\pi}{c}\boldsymbol{j}. \tag{15}$$

The electric field is $\boldsymbol{E}$ and current density is $\boldsymbol{j}$. These equations are usually valid in astrophysical fluids or gases of plasma.

(a) To whom are the names of each of these two equations attributed (after Maxwell's extension of them)? They are named after two very famous physicists.

ANSWER: Equation (14) is Maxwell's extension of Faraday's Law, and equation (15) is his modification of Ampere's Law (Maxwell added the displacement current term).

(b) Do a unit analysis of equation (14) using a characteristic length scale L and time scale t. Also assume a characteristic speed $u = L/t$ that is **non-relativistic**. What can you immediately qualitatively conclude about the magnitudes of B and E?

ANSWER:

$$\frac{E}{L} \sim \frac{1}{c}\frac{B}{t} \tag{16}$$

$$\frac{E}{B} \sim \frac{1}{c}\frac{L}{t} \tag{17}$$

$$\frac{E}{B} \sim \frac{u}{c} \ll 1 \tag{18}$$

We see that the magnetic field is much larger than the electric field for the smallish velocities we are considering.

(c) Do a similar analysis for equation (15). Under the same approximation that you discerned in (b), show that you can safely ignore one of the two terms on the RHS side of that equation.

ANSWER:

$$\begin{aligned}
\frac{B}{L} &\sim \frac{1}{c}\frac{E}{t} + \frac{j}{c} && (19)\\
B &\sim \frac{L}{c}\frac{E}{t} + \frac{j}{c} && (20)\\
B &\sim \frac{u}{c}\frac{E}{B}B + \frac{j}{c} && (21)\\
B &\sim \left(\frac{u}{c}\right)^2 B + \frac{j}{c}, && (22)
\end{aligned}$$

where we've used the result from the analysis in (b) for E/B. We see that the term $c^{-1}\partial_t E$ is *really* small and is known as the *displacement* current.

(d) [***Extra credit:*** **2 points**]. What is the common name of that 'neglected' term?

ANSWER: The *displacement* current.

2. [**35 points**]. Now consider another relationship between the current density and the electromagnetic fields:

$$j = \sigma\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}\right), \tag{23}$$

where σ is the conductivity and $\boldsymbol{v}$ is the velocity of the plasma viewed from a rest frame.

(a) What 'law' is equation (23) usually referred to as? Using equations (14), (15), (23) and any approximations you've made, derive the *induction equation* below solely in terms of the magnetic field:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}. \tag{24}$$

ANSWER: Equation (23) is known as Ohm's Law. Since the thing we want to show has a $\partial_t B$ in it, it makes sense to make good use of equation (14). It is found pretty easily by solving for $\boldsymbol{E}$ in equation (23) and plugging that into equation (14). Then solve for j in equation (15) (neglecting the displacement current as we have shown) and plugging it into the previously derived equation. Now everything is in terms of $\boldsymbol{v}$ and $\boldsymbol{B}$. The last thing is to use the vector identity in equation (45), knowing to use the zero divergence of the magnetic field.

(b) What did you find for the coefficient η? What units must it have? What physical interpretation does the first term on the RHS of equation (24) have? The second term?

ANSWER: $\eta = c^2/4\pi\sigma$. It's units must be L^2/T to match the units of the two other terms (B/t). The first term on the RHS is the advection of the magnetic field due to plasma flows. The second term represents the diffusion or decay of the magnetic field.

(c) Consider a fluid at rest in equation (24). Define a time, $\tau_{\rm d}$, using η and any characteristic scales, as before. It can be shown (for a hydrogen plasma) that $\eta = 10^9\,T^{-3/2}\,\mathrm{m^2\,s^{-1}}$, where T is the temperature, in Kelvin. What would the time $\tau_{\rm d}$ be if we take the magnetic field in a sunspot (assume a plasma temperature a bit less than the surrounding photosphere and a ball-park size for a sunspot)? Comment on the value that you find in terms of the physics and time and length scales. What word do you think the "d" in $\tau_{\rm d}$ might stand for?

ANSWER: The "d" is for diffusion or decay. If we ignore the velocity in the induction equation, we are left with the diffusion equation:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \eta \nabla^2 \boldsymbol{B}. \tag{25}$$

From this expression we can define a diffusion timescale just from unit analysis of the RHS of equation (25).

$$\tau_{\rm diff} \approx L^2/\eta. \tag{26}$$

So this timescale represents a quantity of the temporal solution of equation (25), something like $B(t) \sim \exp(-t/\tau_{\rm d})$. For a sunspot, let's take the temperature to be 5000K, which gives $\eta \approx 3000\,\mathrm{m^2\,s^{-1}}$, and its size to be $L \approx 10^7$m (about Earth sized). This gives a diffusion time of

$$\tau_{\rm diff} \approx 3.5 \times 10^{10}\,\mathrm{s}, \quad \text{or } 1000\,\text{years!} \tag{27}$$

The magnetic field cannot decay away in a sunspot on large length scales due to diffusion: that time scale is just too large. Other factors are responsible for the short lifetime ($\sim$1 month) of sunspots, including diffusion on small scales. Although the magnetic field is still present beyond their actual lifetime, it just gets redistributed.

(d) Astrophysical fluids and plasmas are not generally at rest, however. Therefore, the induction equation is really a struggle between the relative strengths of the two terms. Define an appropriate order-of-magnitude ratio of the two terms on the RHS in equation (24), called the *magnetic Reynolds number*, R_M, using any previous scales and quantities, and show that for typical solar values – try a sunspot again – it is enormous compared to 1 (if it's small, you've probably got your ratio upside-down, so re-define it). What does this imply for the conductivity of typical stars? How would the magnetic Reynolds number change in the corona of the Sun compared to the sunspot (show and describe this with simple hand-wavy arguments)?

ANSWER: The magnetic Reynold's number is given as the ratio of the advective to the diffusive term:

$$R_M = \frac{\nabla \times (\boldsymbol{v} \times \boldsymbol{B})}{\eta \nabla^2 \boldsymbol{B}} \tag{28}$$
$$= \frac{uB/L}{\eta B/L^2} \tag{29}$$
$$= \frac{uL}{\eta} \quad \text{or} \quad \frac{L^2}{\eta \tau}. \tag{30}$$

Taking the same values as before, $L \approx 10^7$m, $\eta \approx 3000\text{m}^2\,\text{s}^{-1}$, and $u \approx 1000$m/s (typical speeds of the plasma near sunspots), we find

$$R_M \approx 10^6 \gg 1. \tag{31}$$

If we look back at the definition of η, we see that it is inversely proportional to σ. Thus, to a good approximation, the plasma is infinitely conducting. It is even truer for the corona, where the length scales are larger and the velocities can be too, and the temperatures are higher (i.e., η smaller).

(e) Since the Reynolds number is so large in most cases, we can safely neglect one of the terms in the induction equation. First write down the resulting induction equation, and explain the physical significance of it in terms of dynamics. (*Hint: For its significance, think about a plasma at rest.*)

ANSWER: We can ignore diffusion for most situations in stars, so the induction equation is

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}). \tag{32}$$

As mentioned before, this is an advection term. What it says is that the field gets swept away by the plasma flows and moves with it. This sometimes is referred to as the field being "frozen in" to the plasma. If there are no flows, then the magnetic field is constant in time – this implies that a flow affects the magnetic field.

(f) [***Extra credit*: 3 points**]. Current 3D numerical simulations of convection in stars are only able to employ Reynolds numbers of roughly $\sim 10^3$ or so, much smaller than what we've seen to be the typical values. Based on your derivations, what factors may be limiting pushing this value higher in these simulations? Can you think of any numerical schemes or tricks to get around this obstacle to make the equations in the simulations more physical?

ANSWER: First consider that large Reynolds numbers imply large length scales, velocities, etc. But numerical simulations use spatial and time derivatives that require very fine meshes for accuracy. It is very hard to fulfill both requirements simultaneously, and so compromises are made. Intermediate Reynolds numbers are used, but this leads to other problems. One can't take

an intermediate number and then ignore the diffusion term, but this is common. Therefore, what codes use are "artificial diffusivities", which damp out unwanted modes or instabilities without having to solve a full self-consistent induction equation. These are phenomenological terms that are added for convenience.

Congratulations! You've just (hopefully) derived a good deal of ideal magnetohydrodynamics.

III Applications

1. [**25 points**]. We've been talking about the magnetic field so much, let's find out if it's actually important for stars after all! The above MHD equations need to be coupled to an equation of motion to be useful. Let's define it as

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla} P + \boldsymbol{g} + \frac{1}{\rho c}\boldsymbol{j} \times \boldsymbol{B}, \tag{33}$$

where P is the gas pressure, ρ is mass density, and $\boldsymbol{g}$ is the acceleration due to gravity.

(a) What is the term $c^{-1}\boldsymbol{j} \times \boldsymbol{B}$ commonly called?

ANSWER: It is clear that this is a force and it is known as the Lorentz force (sometimes there is also an electric field in there.)

(b) Consider a stationary gas where the effects of gravity are not significant over the length scales involved compared to the other terms. Using equation (33) and any other ones you may need, define and derive what's called the plasma β parameter:

$$\beta = \frac{4\pi P}{B^2}, \tag{34}$$

where $B = |\boldsymbol{B}|$.

ANSWER: After ignoring the terms we need to we have

$$\boldsymbol{\nabla} P = \frac{1}{c}\boldsymbol{j} \times \boldsymbol{B}. \tag{35}$$

From Ampere's Law,

$$\boldsymbol{j} = \frac{c}{4\pi}\boldsymbol{\nabla} \times \boldsymbol{B}. \tag{36}$$

Then

$$\boldsymbol{\nabla} P = \frac{1}{4\pi}(\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B}, \tag{37}$$

and taking a characteristic length scale over which the gradients don't vary much:

$$\frac{P}{L} = \frac{B^2}{4\pi L}, \tag{38}$$

$$\textbf{Let} \quad \beta = \frac{4\pi P}{B^2}. \tag{39}$$

(c) The plasma β is an *approximate* measure of the relative strength of the gas pressure to the magnetic pressure. Let's find some values of it for the Sun. Consider the Sun to be pure, fully ionized hydrogen and an ideal gas. Show that the plasma β parameter is

$$\beta \approx 7 \times 10^{-15}\, n_{\mathrm{e}} T B^{-2}, \tag{40}$$

where n_{e} is the electron number density, and B is usually given in units of gauss (G).

ANSWER: So $P = \rho RT/\mu$. By ignoring the electron mass, we find that $\mu = 0.5$ using the appendix, so $P = 2\rho RT$. We have so far

$$\beta = \frac{8\pi}{B^2}\rho RT. \tag{41}$$

The next step is to realize that the mass density ρ is just the particle number density times the mass, $\rho = nm$. The number of particles, because of full ionization, is $n_{\mathrm{e}} + n_{\mathrm{p}} = 2n_{\mathrm{e}}$, and we can use the mass of the proton. So $\rho = 2n_{\mathrm{e}}m_{\mathrm{p}}$. Collecting now we have

$$\begin{aligned} \beta &= 16\pi R m_{\mathrm{p}} n_{\mathrm{e}} T B^{-2} && (42) \\ &= 16\pi(8.3 \times 10^{7})(1.67 \times 10^{-24}) n_{\mathrm{e}} T B^{-2} && (43) \\ &\approx 7 \times 10^{-15}\, n_{\mathrm{e}} T B^{-2}. && (44) \end{aligned}$$

(d) Use the above equation to describe, for a star like the Sun, when we can ignore magnetic fields or when magnetic fields dominate. Consult Figures 4 and 5 for some values you may need. Use your best guesses for all the quantities you do not have at hand. Consider specifically β in the (*i*) radiation zone, (*ii*) the 'quiet' photosphere, (*iii*) a sunspot, and (*iv*) the corona. Explain why, physically, these values of β make sense to you, as you work through each region.

ANSWER: (*i*) For the radiation zone let's consider the halfway point of the solar radius. A sensible temperature is about 7 million K. The main point here is that there is no known magnetic field to speak of. The electron density, from the figure, is 10^{24}. We can take 1 gauss for the B field. This gives $\beta \approx 10^{16}$. Huge! This makes sense since the plasma pressure just dominates anywhere inside the Sun. Magnetic fields are confined, if there are any, to thin flux tubes, unable to expand. (*ii*) At the surface, we take a temperature of 6000K, $n_{\mathrm{e}} \approx 10^{16}$, and a quiet-Sun field over normal length scales is usually about 100 G. This gives $\beta \approx 40$. The plasma still dominates, but the magnetic field has more freedom. (*iii*) In a sunspot, let's take the temperature to be 5000 K and the field to be 2000 G. Keeping the density the same as the surrounding plasma (even though it's likely less since the spot is "evacuated"), we find $\beta \lesssim 0.1$. One must expect here an answer less than one, otherwise sunspots would not hold their structure. (*iv*) In the corona, from Figure 5 we can take a temperature of 2 million degrees and a density of 10^{8}. The magnetic field is not so strong, relatively, so we'll take $B = 10$ G. This gives $\beta \approx 0.014$. Again, we know the plasma moves

with the magnetic field in the corona (think coronal loops), so the answer should be less than one. It should be quite less than one too. It should be noted however that there are regions of the corona where β is greater than 1, and this is close to the far extent of it where the solar wind is accelerated.

Appendix: Useful Items

- For any vector $\boldsymbol{A}$:

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) - (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\boldsymbol{A} \tag{45}$$

- The ideal gas law is

$$P = \frac{\rho R T}{\mu}, \tag{46}$$

where $R = 8.3 \times 10^7\,\mathrm{cm^2 s^{-2} K^{-1}}$ is the gas constant, ρ is the mass density, and μ is related to the mean molecular weight $\overline{m}$ by

$$\overline{m} = \frac{n_\mathrm{p} m_\mathrm{p} + n_\mathrm{e} m_\mathrm{e}}{n_\mathrm{p} + n_\mathrm{e}} = \mu m_\mathrm{p}. \tag{47}$$

The ns are number particle densities, the ms are masses, and the subscripts p and e denote proton and electron, respectively.

- Mass of a proton $= 1.67 \times 10^{-24}$ g.

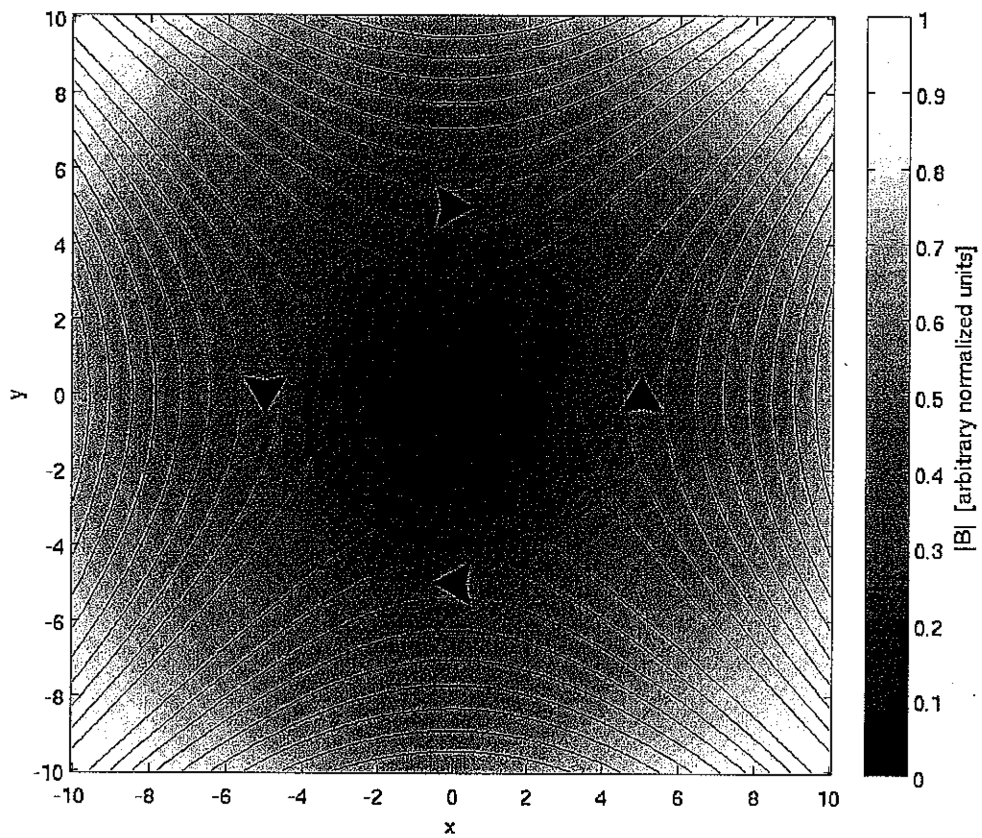

Figure 1: Field lines and directions for $\boldsymbol{B} = (y, x, 0)$. The background gray scale is the magnitude of the magnetic field, $|\boldsymbol{B}|$, normalized to unity. See problem I.1

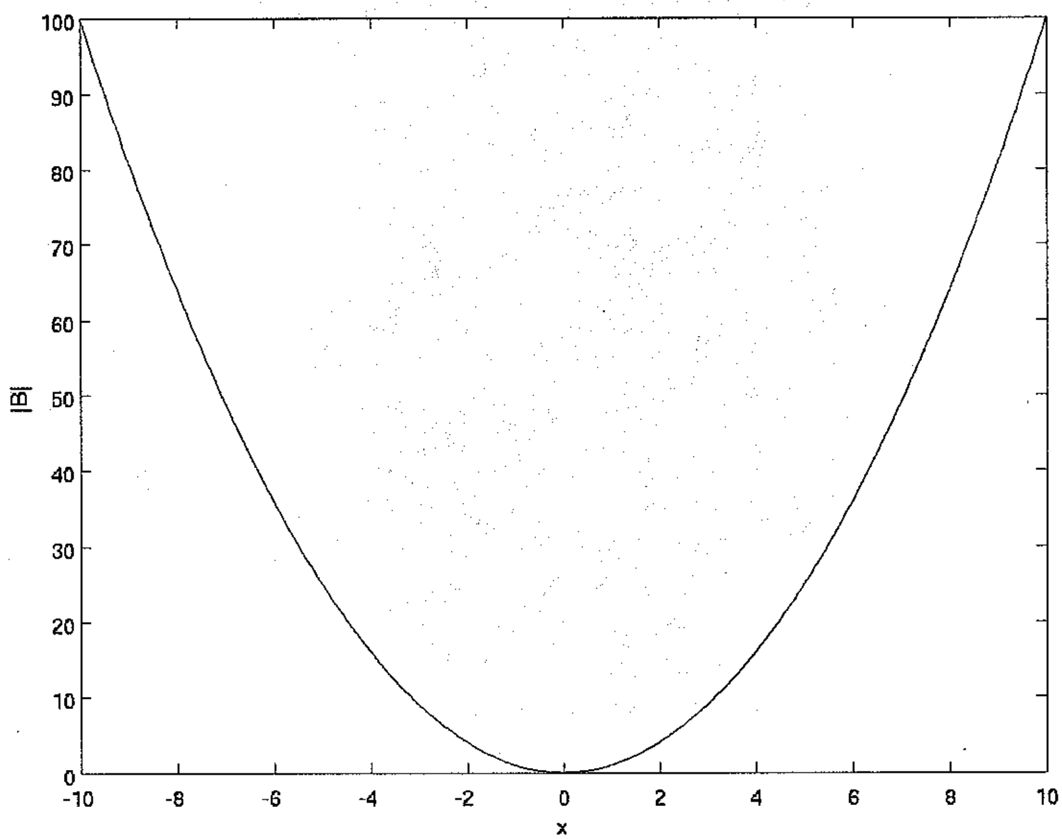

Figure 2: Magnitude of magnetic field for $\boldsymbol{B} = (0, x^2, 0)$. See problem I.2

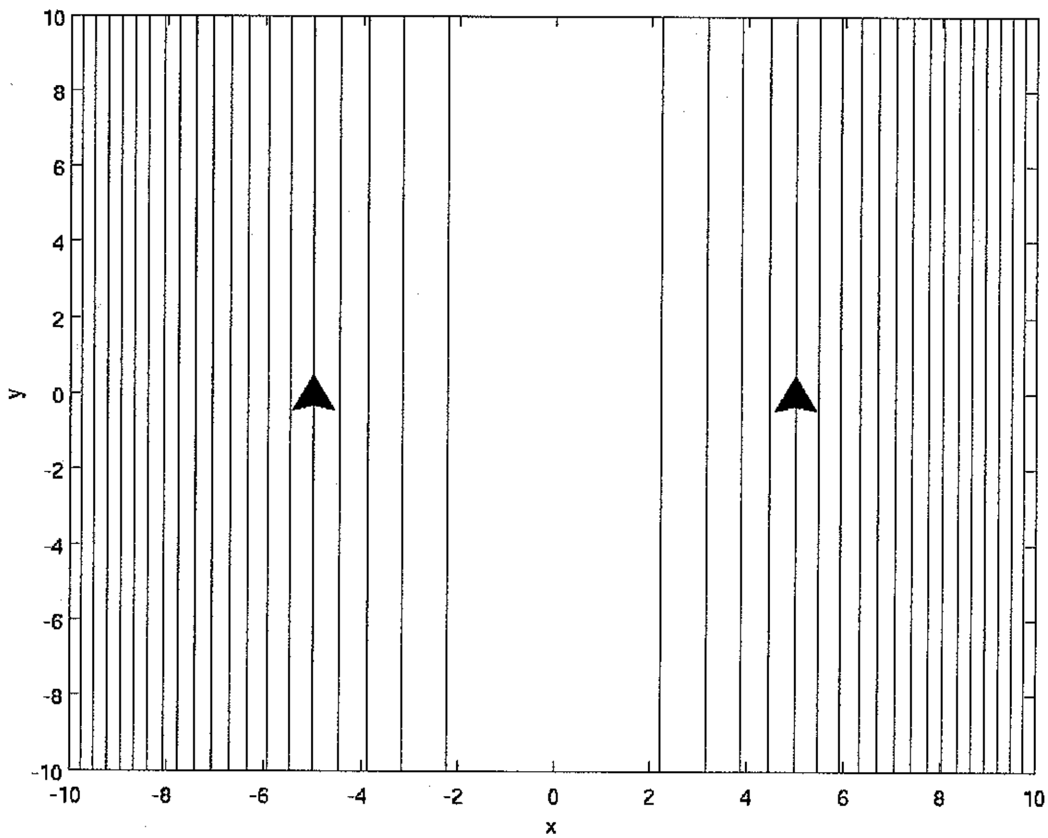

Figure 3: Field lines and directions for $\boldsymbol{B} = (0, x^2, 0)$. See problem I.2

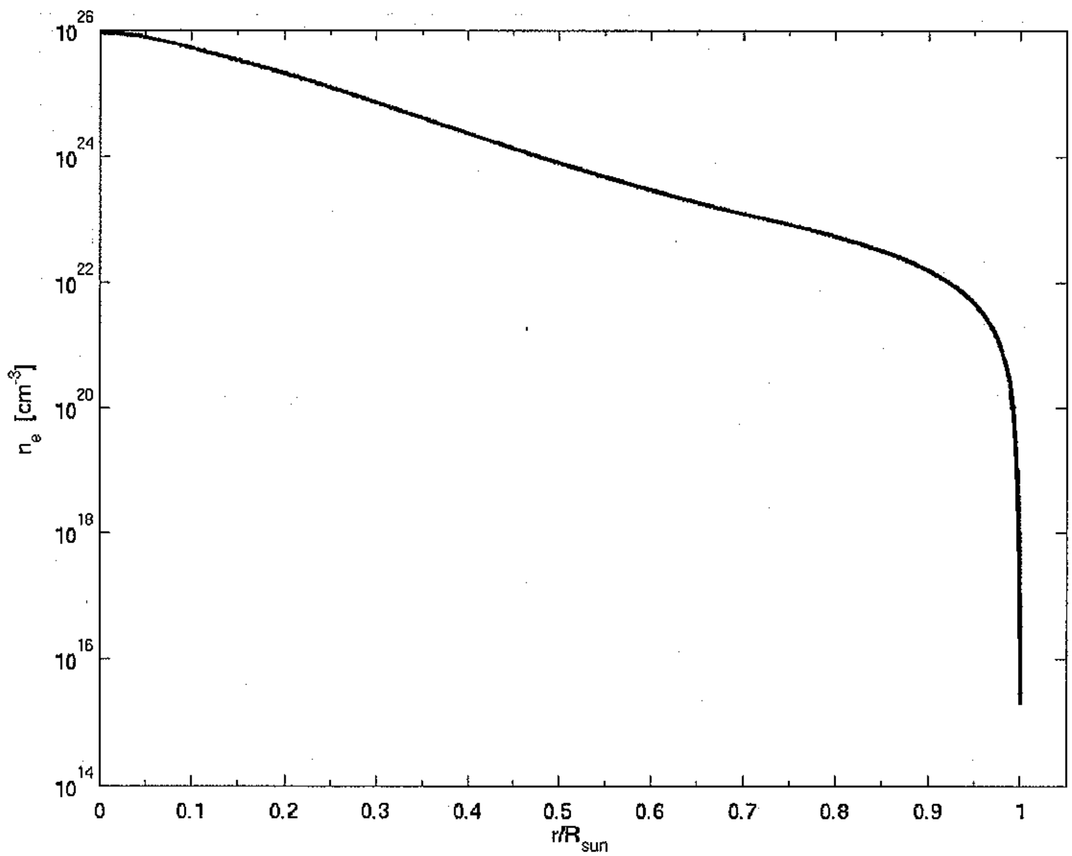

Figure 4: Electron number density versus fractional radius from a model of the Sun. See Problem III.1d.

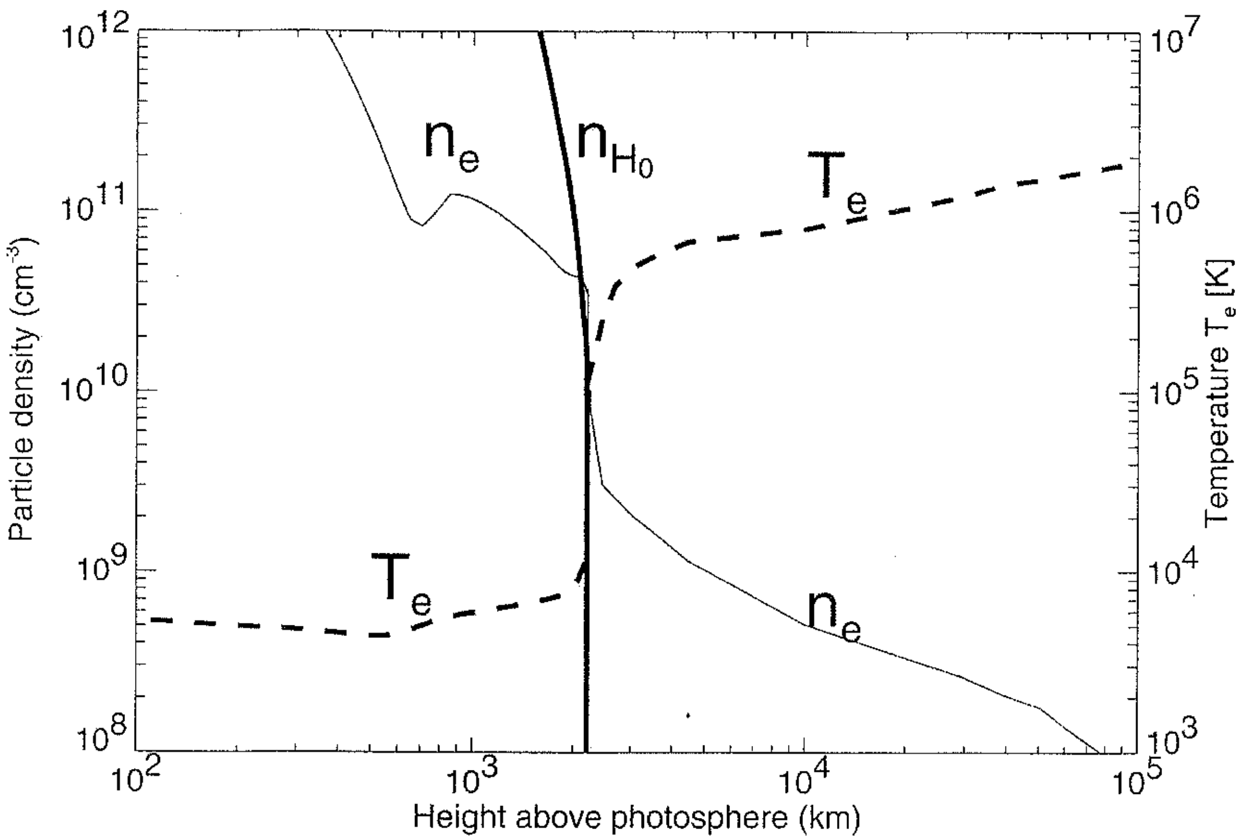

Figure 5: Electron number density and temperature versus height in the solar atmosphere. Ignore n_{H_0} and the thick vertical line. From Aschwanden 2006. See Problem III.1d.