Definitions, Methods, and Tools

- Dynamical time, Circular velocity, Rotational velocity
- Virial theorem, Jeans equations
- Two-body scattering
- Dynamical friction

Circular velocity V_c is defined as: $V_c^2/r = g(r)$. where $g(r)$ is gravitational acceleration. For spherical system

$$g(r) = \frac{GM(r)}{r^2}, \quad M(r) = 4\pi\int_0^r \rho(r)r^2 dr. \tag{1}$$

Circular velocity is the velocity of an object moving on a circular orbit under the action of the force of gravity. It does not depend on the rotation of the object. It is another way of measuring mass distribution.

Dynamical time. First consider a homogeneous sphere with constant density $\rho(r) = const$. Take a particles at rest at a distance r from the center and drop it. Equation of motion of the particle is:

$$\frac{d^2r}{dt^2} = -\frac{GM(r)}{r^2} = -\frac{4\pi G}{3}\rho r \tag{2}$$

This is equation of harmonic oscillator with frequency ω : where $\omega^2 = \frac{4\pi}{3}G\rho$. Period of oscillation is equal to

$$T = \frac{2\pi}{\omega} = \sqrt{\frac{3\pi}{G\rho}}. \tag{3}$$

Now we *define* the dynamical time as 1/4 of the period:

$$T_{\rm dyn} \equiv \frac{T}{4} = \sqrt{\frac{3\pi}{16G\rho}}. \tag{4}$$

For a sphere with constant density the dynamical time does not depend on initial radius. It depends only on the density ρ. This is not true for other density profiles. We typically expect that $T_{\rm dyn}$ is shorter for orbits closer to the center where density is larger. We treat the dynamical time as a scale, not as an accurate value.

Other related quantities: **Free-fall** time. We get an estimate of this allowing the sphere in the previous example to collapse. This is the same as having all the mass at the center. Typical estimate:

$$T_{\rm freefall} = T_{\rm dyn}/\sqrt(2). \tag{5}$$

Crossing time: If R is typical radius of a system and V is typical velocity, then crossing time is:

$$T_{\rm cross} \approx \frac{R}{V}. \tag{6}$$

Virial theorem. We have an isolated system of N objects. Masses, coordinates and velocities are given: $m_i, \vec{r_i}, \vec{V_i}$. We are looking for a relation between the total kinetic energy and the total potential energy. The energies are:

$$\mathcal{K} = \frac{1}{2}\sum_i m_i(\dot{x_i}^2 + \dot{y_i}^2 + \dot{z_i}^2), \quad \mathcal{W} = -\frac{1}{2}G\sum_{i,j}\frac{m_i m_j}{|\vec{r_i} - \vec{r_j}|} \tag{7}$$

We start with defining the moment of inertia:

$$\mathcal{I} = \sum_{i=1}^{N} m_i[x_i^2 + y_i^2 + z_i^2]. \tag{8}$$

We will differentiate $\mathcal{I}$ twice with respect to time t and then use equations of motion. At some moment we use the Newton's law of gravity. At the end of calculations we find:

$$\frac{d^2\mathcal{I}}{dt^2} = 2\mathcal{K} + \mathcal{W}. \tag{9}$$

This is the virial theorem. Typically it is applied with the assumption that the $d^2\mathcal{I}/dt^2 = 0$ implying a *stationary system*.

We can use the virial theorem to estimate the mass of a stationary isolated system. We introduce the gravitational radius r_g of a system:

$$\mathcal{W} = -\frac{GM^2}{r_g} \tag{10}$$

Note that here we do not assume anything about the system. We do not assume that is spherical or its density is smooth; no assumptions. This equation simply states that potential energy scales as M^2. Details of the mass distribution are hidden in r_g. For a system with equal-mass points we have:

$$\frac{1}{r_g} = \langle \frac{1}{r_{ij}} \rangle. \tag{11}$$

Kinetic energy can be parameterized as

$$\mathcal{K} = -\frac{M\langle V^2 \rangle}{2}, \tag{12}$$

where $\langle V^2 \rangle$ is the velocity dispersion. Again, no assumptions here: this is always valid. Assuming that the line-of-sight velocity dispersion is 1/3 of the 3d velocity dispersion, we get the following estimate of the mass:

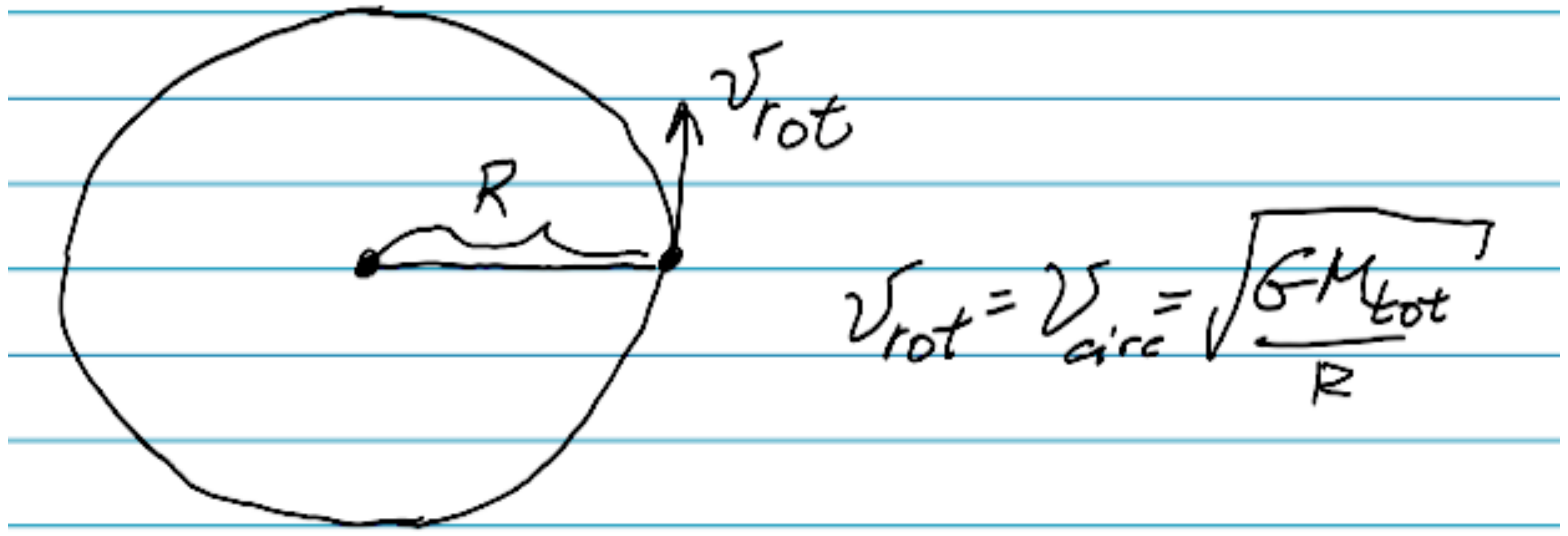
$$M = \frac{3\langle V_{los}^2 \rangle r_g}{G}. \tag{13}$$

Here we specifically assumed that the observed rms velocities are equal to the rms velocities of all material in the system, which may not be correct. However, experience with different systems (either analytical models or numerical simulations) shows that this is not a bad approximation. The main problem is to estimate r_g. There are a number of problems with the estimate. First, r_g typically is a radius, which is dominated by large-distance pairs. Each pair has a relatively small contribution, but there are many of them and they dominate the final result. Because observationally it is more difficult to measure objects in outer radii of a system, we are susceptible to numerous observational complications (e.g, projection or non-equilibrium effects). Second, spatial distribution of observed objects may be very different as compared with, say dark matter.

It is remarkable that the virial theorem still gives sensible estimates of masses. It is generally believed that it gives estimates within a factor of 2 of the real values.

Jeans equations:. We would like to have methods to estimate masses of galaxies. The virial theorem is not a useful tool for that for different reasons. First, virial radius of galaxies is substantially larger than the size of regions where we can reliably measure velocities. For example, for our Galaxy the virial radius is about 250 kpc while the optical radius is about 10-20 kpc. For Milky Way we can use motion of remote globular clusters and satellite galaxies. However, this information is not available for other galaxies. Second, the virial theorem gives us only one mass: mass of the whole system, and we would like to measure density profile, not just one point.

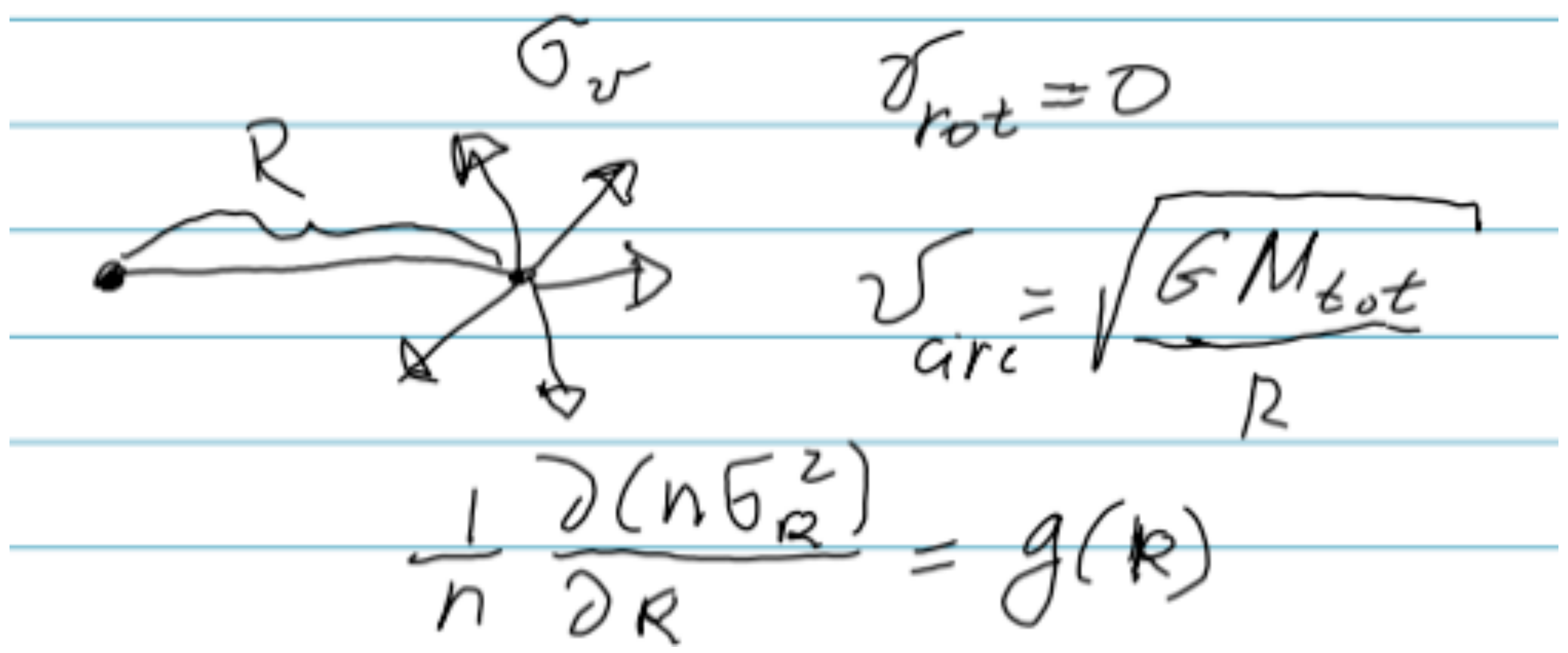
The main path to address this issue is to start with the Boltzmann equations then assume a stationary system, and derive Jeans equations that relate stellar kinematics (different components of velocities) with the mass distribution. Here we just use those equations without a rigorous derivation.

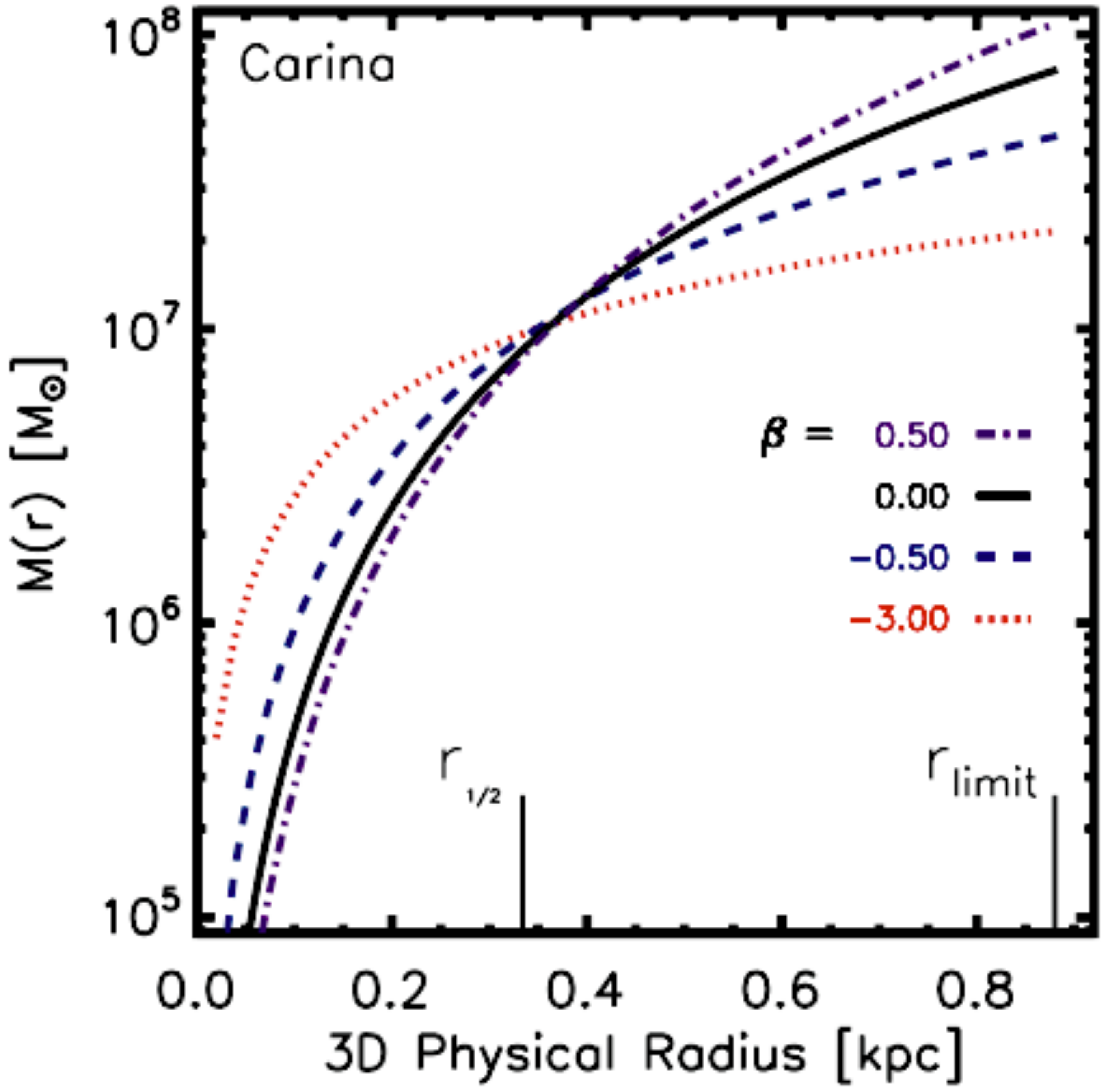


Start with the simplest case and then add more realistic features. Suppose we have a system where all stars move on pure circular orbits at different distances from the center. There are no other motions – just circular velocities. However, there is a distribution of mass $M(r)$ so that the gravitational acceleration g changes with radius. For simplicity assume that the mass distribution is spherical. Now we can relate the rotational velocity with the mass $M(r)$:

$$\frac{V_{\rm rot}^2}{r} = g = \frac{GM(r)}{r^2} \rightarrow V_{\rm rot} = \sqrt{\frac{GM(r)}{r}}. \qquad (14)$$

In this case rotational velocity $V_{\rm rot}$ is equal to the circular velocity. We measure the rotational velocities in observations and recover the distribution of mass $M(r)$. Note that here the mass is the *total* mass of all components (visible or not) while the rotation is measured using some subsample of the stars in the system.





Cumulative mass profile of Carina dSph galaxy (satellite of the Milky Way) using four different constant velocity anisotropies

Another case: no rotation, isotropic velocities. In this case there are only random vel ocities and no systematic rotation. At each point in space velocity distribution is isotropic with all three velocity dispersions equal: $\sigma_r = \sigma_\phi = \sigma_\theta$. Jeans equations take the form of familiar equation of hydrostatic equilibrium:

$$\frac{1}{n}\frac{\partial(n\sigma_r^2)}{\partial r} = -\frac{GM(r)}{r^2}, \qquad (15)$$

where $n(r)$ is the number-density of objects at distance r from the center. This gives us way to find mass (and density) profile for the system: measure the number-density of a stellar population and their rms velocities. Then use the equation to find $M(r)$.

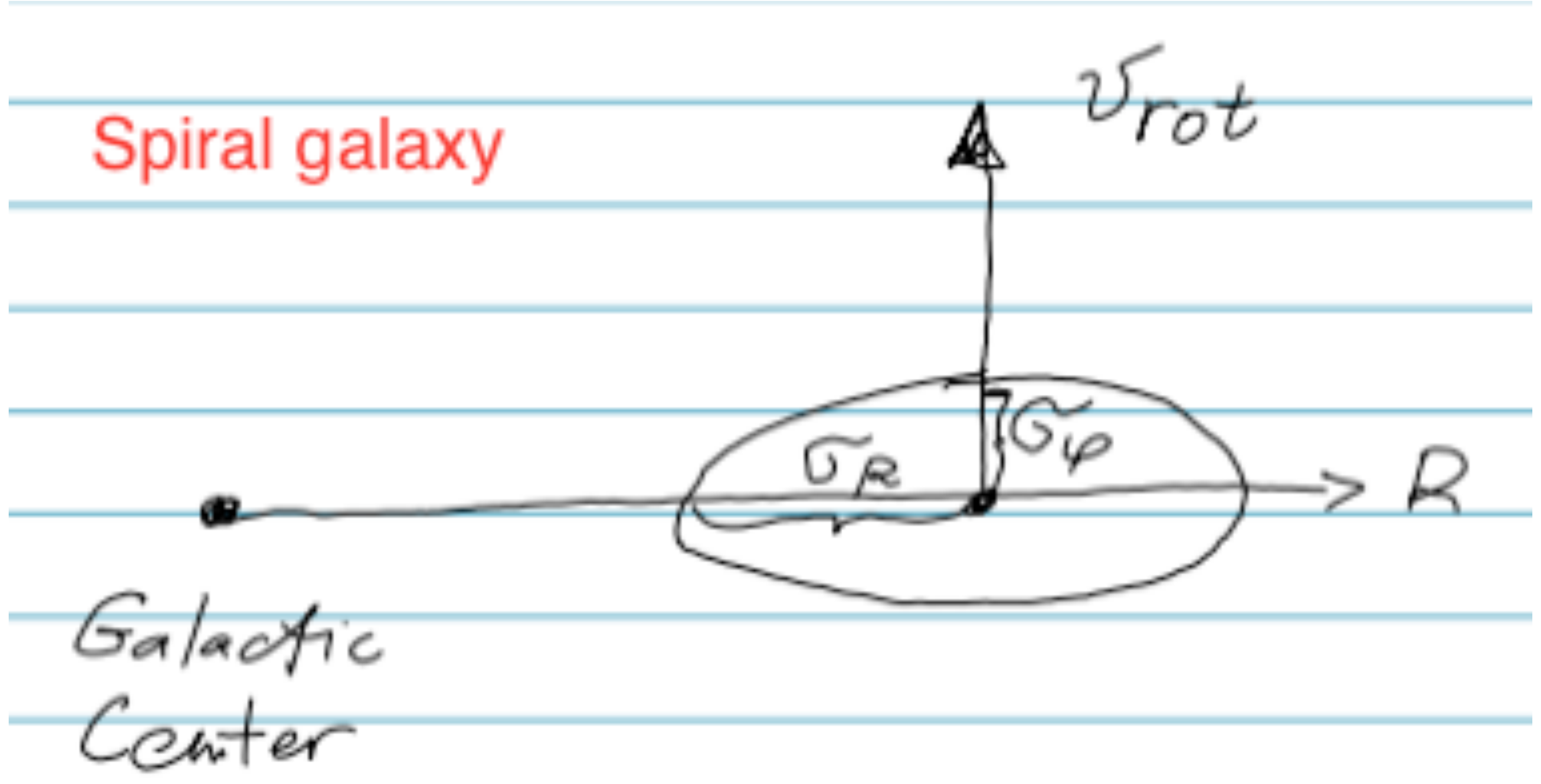
In more general case there is a velocity anisotropy:

$$\beta \equiv 1 - \frac{\sigma_\phi^2 + \sigma_\theta^2}{2\sigma_r^2}. \qquad (16)$$

In this case the Jeans equation is:

$$\frac{1}{n}\frac{\partial(n\sigma_r^2)}{\partial r} + \frac{2\beta\sigma_r^2}{r} = -\frac{GM(r)}{r^2}, \qquad (17)$$

In order to apply this equation one needs to make an assumption regarding the velocity anisotropy β. The eq. 17 is typically used for elliptical galaxies and dwarf spheroidal

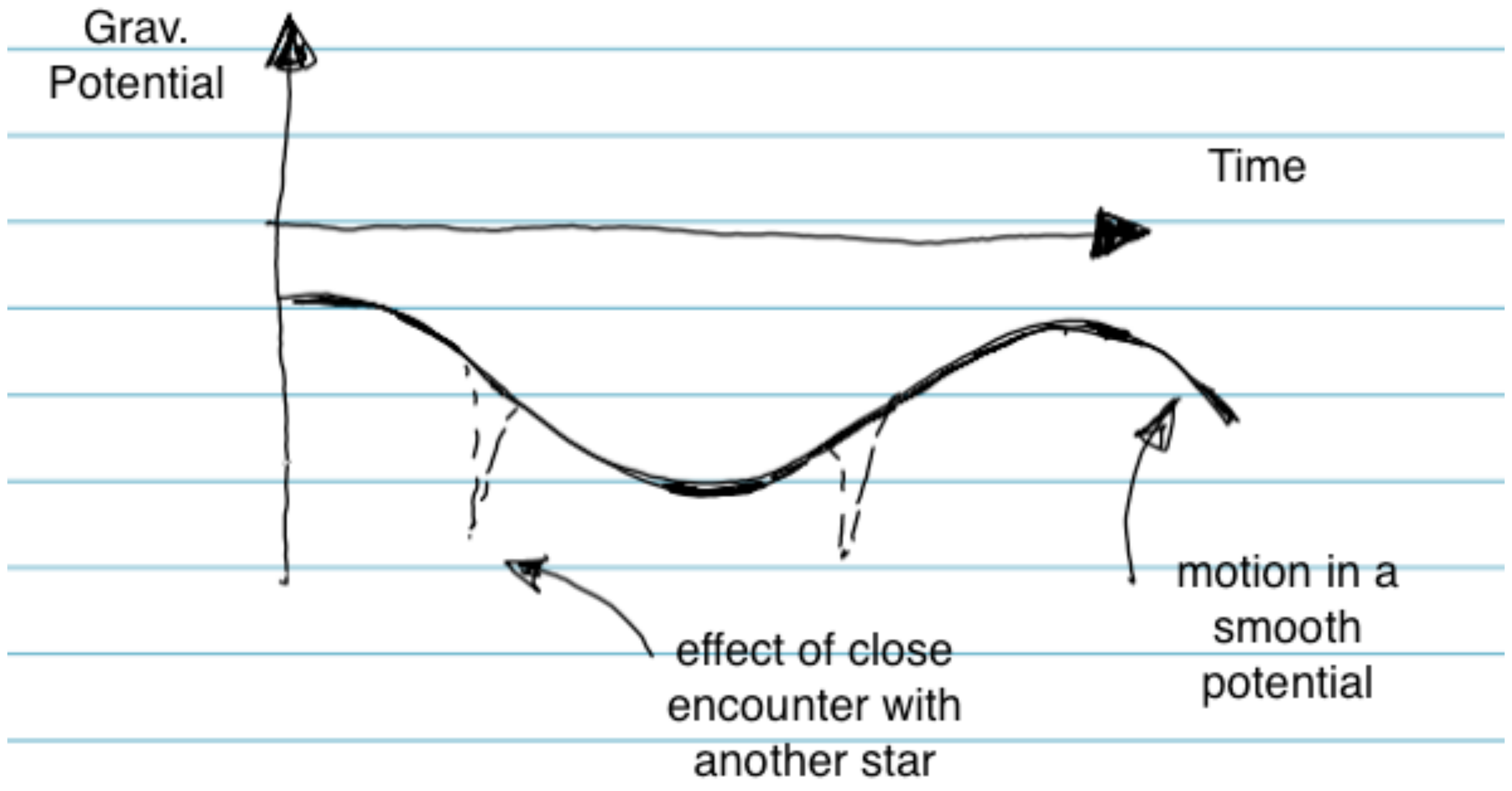


For spiral galaxies with fast rotation the Jeans equation can be written slightly differently. Quantities involved: $V_{\rm rot}$ = rotational velocity. σ_ϕ^2 = velocity dispersion in the direction of rotation. σ_r^2 = velocity dispersion in the radial direction. $n(r)$ = number-density of stars with measured velocities. $V_c^2 = g(r)r$ - circular velocity. The Jeans equation is:

$$V_{\rm rot}^2 = V_c^2 + (\sigma_r^2 - \sigma_\phi^2) + \frac{r}{n}\frac{\partial(n\sigma_r^2)}{\partial r}. \tag{18}$$

Note that we expect that the rotational velocity be smaller than the circular velocity $V_{\rm rot}^2 \leqslant V_c^2$ because the radial velocity dispersion is larger than the tangential velocity dispersion and the last term is negative. Again, in spite of the fact that we measure velocities of some observable component (say bright stars), the recovered circular velocity and, thus mass is for **all** mass.

How important are effects of discreteness: the fact that there are individual starts, not just a smooth density distribution



$$t_{\rm relax} = 0.34 \frac{\sigma^3}{G^2 m \rho \ln \Lambda},$$

Two-body relaxation plays a central role in the evolution of most clusters considered here. The local relaxation time is (Spitzer 1987)

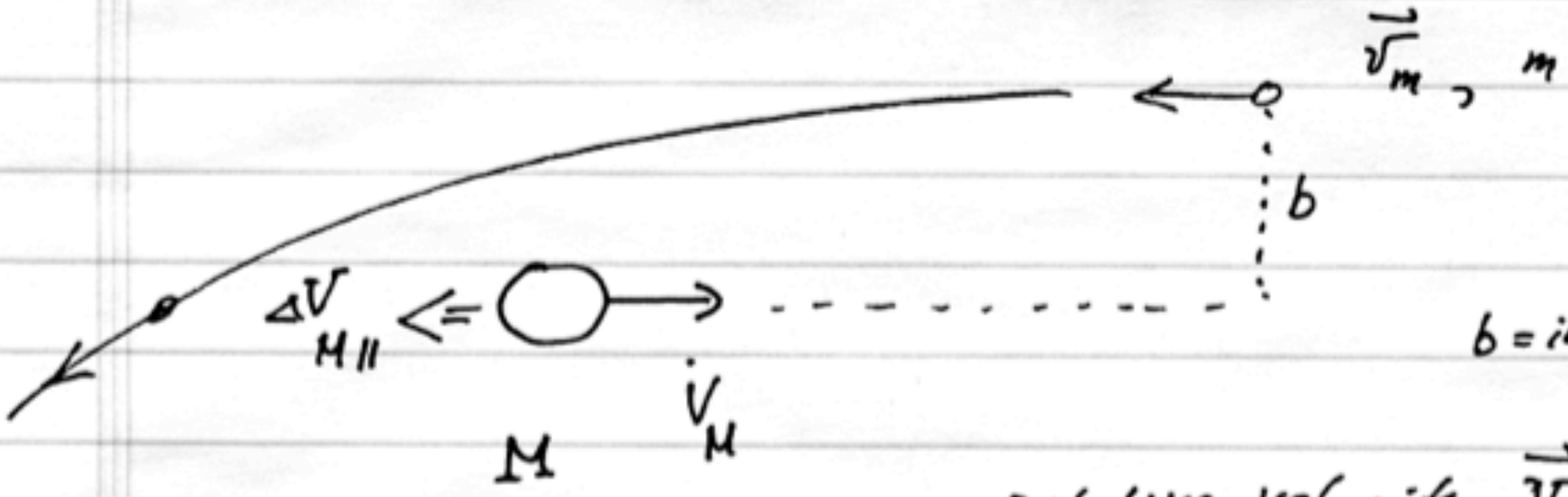
$$t_{\rm rlx} = 0.339 \frac{\sigma_{\rm v}^3}{G^2 \ln(\gamma_{\rm c} N_*)\, n \langle M_* \rangle^2} \simeq 4.8 \times 10^7 \,{\rm yr} \times \frac{10}{\ln(\gamma_{\rm c} N_*)} \left(\frac{\sigma_{\rm v}}{30\,{\rm km\,s^{-1}}} \right)^3 \frac{10^6\,{\rm pc^{-3}}}{n} \left(\frac{\langle M_* \rangle}{{\rm M_\odot}} \right)^{-2}, \quad (4)$$

where $\langle M_* \rangle$ is the average stellar mass and N_* the total number of stars. For systems with a broad stellar mass spectrum, we use $\gamma_{\rm c} = 0.01$ in the Coulomb logarithm

Object	Number N mass m	Radius R	rms Velocity v	Dynamical Time t_{dyn}	Relaxation time t_{rel}	age
Open clusters	~ 250 $1 M_\odot$	2pc	~ 1 km/s	$2 \cdot 10^6$ yrs	$5 \cdot 10^7$ yrs	$7 \cdot 10^8$ (Hyades) 10^7 yrs (χ Per) $5.5 \cdot 10^9$ yr (NGC 188)
Globular Clusters	$\sim 10^6$ $1 M_\odot$	10pc	~ 10 km/s	10^6 yrs	$2 \cdot 10^9$ yrs	~ 13 Gyrs
Solar Neighborhood	$0.1\,pc^{-3}$ $1 M_\odot$	10kpc	20 km/s (220 km/s)	$2 \cdot 10^8$ yrs	$2 \cdot 10^{13}$ yrs	~ 13 Gyrs
Galaxy Groups	20 $10^{11}-10^{12} M_\odot$	1 Mpc	150 km/s	$5 \cdot 10^9$ yrs	$2.7 \cdot 10^{11}$ yrs $2.7 \cdot 10^9$ yrs	~ 10 Gyrs
Galaxy Clusters	10^3 $10^{11}-10^{12} M_\odot$	2Mpc	~ 1000 km/s	$2 \cdot 10^9$ yrs	10^{12} yrs 10^{10} yrs	~ 10 Gyrs

Dynamical Friction

Object with mass M moves through a field of small objects. Mass of each small object is m



b = impact parameter

Relative velocity $\vec{V} = \vec{v}_m - \vec{v}_M$

$$b_{min} = \frac{G(m+M)}{V^2}$$

Result of a single collision gives

$$\left| \Delta V_{M\parallel} \right| = \frac{2m}{m+M} \cdot V \cdot \frac{1}{1 + \left(\frac{b}{b_{min}}\right)^2}$$

Cumulative effect of many collisions is a sum of individual contributions:

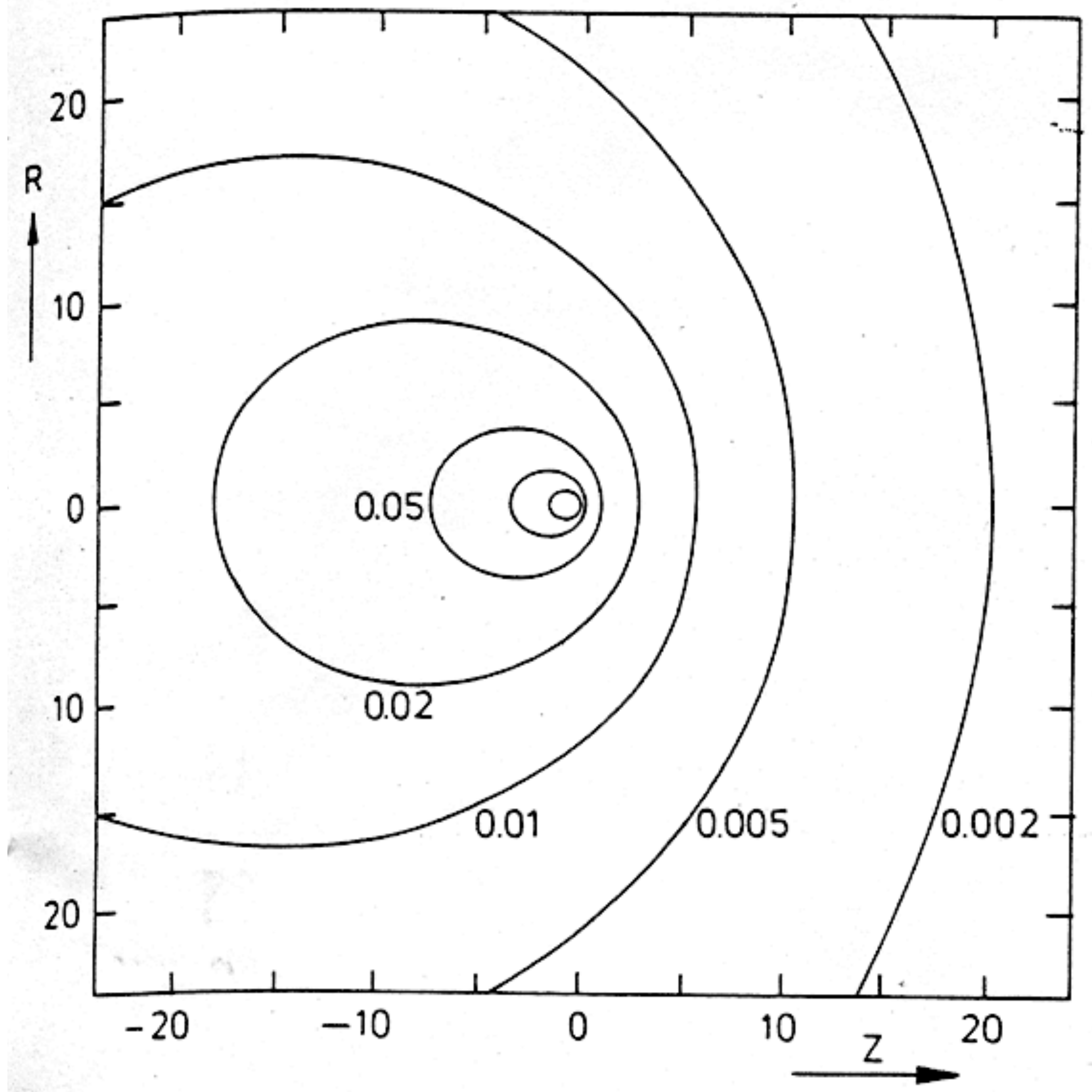


Figure 7-3. A mass tra
from left to right at spee
v through a homogeneou
Maxwellian distribution of
with one-dimensional disp
$\sigma = v$. Deflection of the
by the mass enhances the
density downstream more
upstream. Contours of ec
stellar density are labeled
the corresponding fractior
density enhancement. (Fr
Mulder 1983.)

$$\frac{d\mathbf{v}_M}{dt} = -\frac{4\pi \ln \Lambda G^2 \rho M}{v_M^3}\left[\operatorname{erf}(X) - \frac{2X}{\sqrt{\pi}} e^{-X^2}\right] \mathbf{v}_M$$

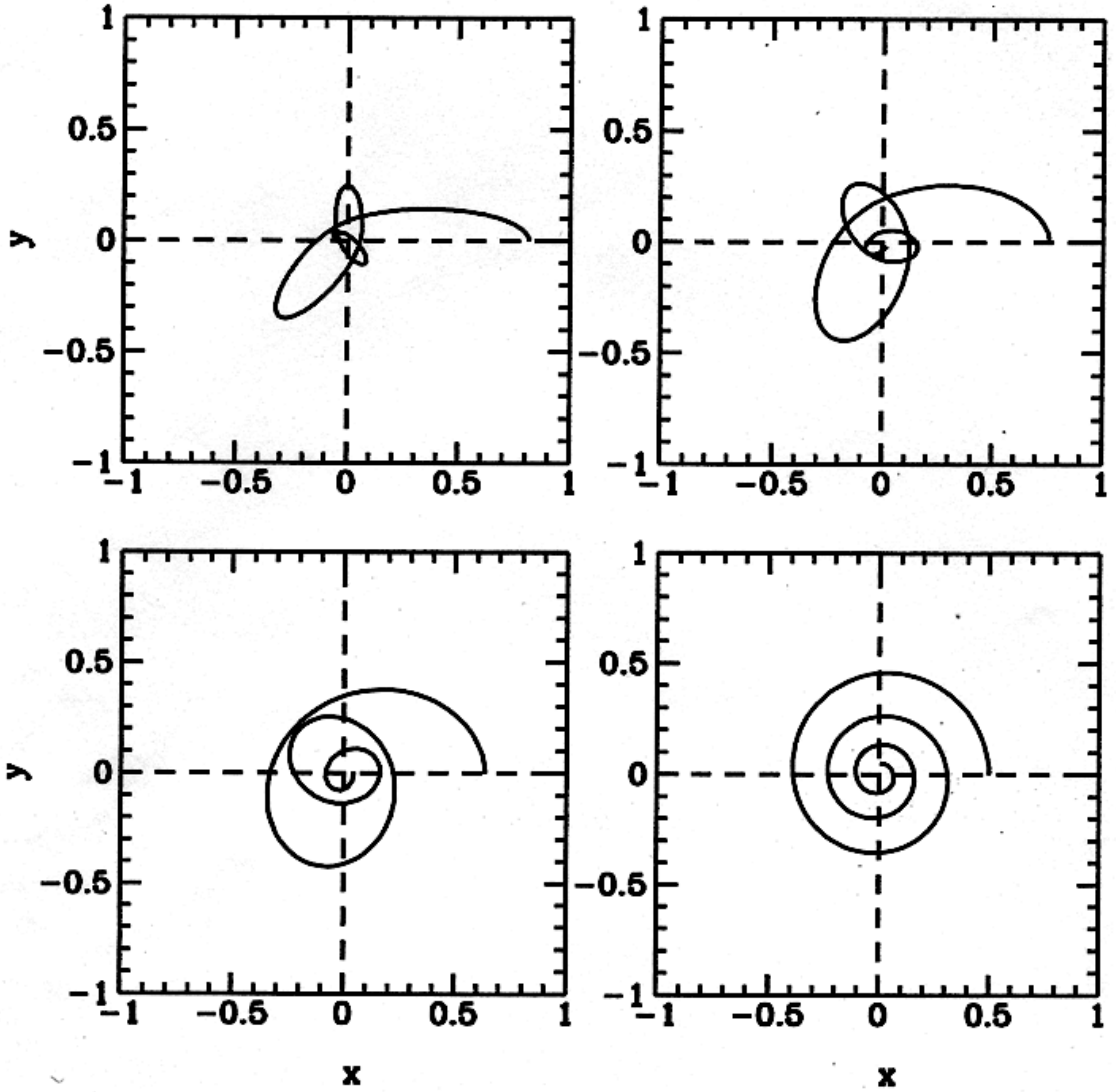


FIG. 1.—Collection of orbits in the plane (x, y) computed within TLR, for $r_{cir}/r_t = 0.5$ and initial eccentricities $e_{orb} = 0.8$ (*top left*), 0.6 (*top right*), 0.3 (*bottom left*), and 0 (*bottom right*). Length is in units of r_t.

⇒ For large v_M and $M \gg m$ the effect scales as

$$\frac{dv_M}{dt} \propto \frac{\rho M}{v_M^2}$$, where $\rho = n_0 m =$ = density of small objects

⇒ For quasi circular orbit in system with $\rho \propto r^{-2}$

$$t_{fric} \simeq \frac{1.17\, r_{init}^2\, v_{circ}}{G M \ln\Lambda} \simeq$$

$$\simeq \frac{2.6 \cdot 10^{11}\,\text{yrs}}{\ln\Lambda} \left(\frac{r_{init}}{2\,\text{kpc}}\right)^2 \left(\frac{v_{circ}}{250\,\text{km/s}}\right) \left(\frac{10^6 M_\odot}{M}\right)$$

⇒ For non-circular orbits the eccentricity of the orbit does not decrease with time