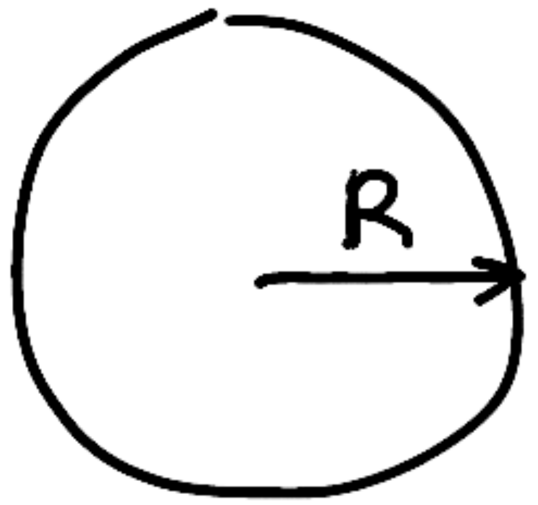
Friedmann Equation: Newtonian derivation

Consider a sphere, which expands in a homogeneous Universe. For non-relativistic particles the mass inside the sphere is constant. We need to find how the radius of the sphere changes with time. Later we will add corrections due to effects of GR.



$$\frac{d^2R}{dt^2} = -\frac{GM(R)}{R^2} = -\frac{4\pi}{3} G\rho R \qquad (*)$$

Use the fact that mass inside comoving radius is preserved:

$$M = \frac{4\pi}{3}\rho R^3 \rightarrow R^3\frac{d\rho}{dt} + 3\rho R^2\frac{dR}{dt} = 0 \Rightarrow$$

$$\frac{d\rho}{dt} + 3H\rho = 0$$

Multiply eq(*) by $2\dot{R}$ we get $2\dot{R}\ddot{R} = -\frac{8\pi G}{3}\rho R\dot{R}$

The l.h.s. is a full derivative: $(\dot{R}^2)^{\cdot}$

The r.h.s. also can be written as a full derivative: $\rho R\dot{R} = -\rho R^3 \frac{d}{dt}\left(\frac{1}{R}\right)$

Now, we can integrate the equation to get:

$$(\dot{R})^2 = \frac{8\pi G}{3}\rho R^2 + A, \qquad A = const$$

Divide both sides by R^2 and use $\frac{\dot{R}}{R} = \frac{\dot{a}}{a} = H$

$$(**)\quad H^2 = \frac{8\pi G}{3}\rho + \frac{B}{a^2}, \qquad \text{here } B = A\left(\frac{a}{R}\right)^2 = Const$$

Let's first find the constant B. At a=1, $H=H_0$, and introduce a new constant:

$$\rho_0 = \Omega_0 \frac{3H_0^2}{8\pi G}$$

Put these relations into eq(**) and find that $B = H_0^2(1-\Omega_0)$

Thus, $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{H_0^2(1-\Omega_0)}{a^2} \quad \Big|\ P=0,\ \Lambda=0$

Because for non-relativistic particles mass in a comoving volume is preserved, we can re-write the Fieadmann equation in a different form:

$$\left(\frac{H}{H_0}\right)^2 = \frac{\Omega_0}{a^3} + \frac{1-\Omega_0}{a^2} \quad \Big\| \quad \rho = \frac{\rho_0}{a^3} = \frac{\Omega_0}{a^3}\frac{3H_0^2}{8\pi G}$$

Critical density: $\rho_{cr}(t) \equiv \dfrac{3H^2}{8\pi G}$

Special case: $\Omega_0 = 1 \Rightarrow H \propto a^{-3/2}$

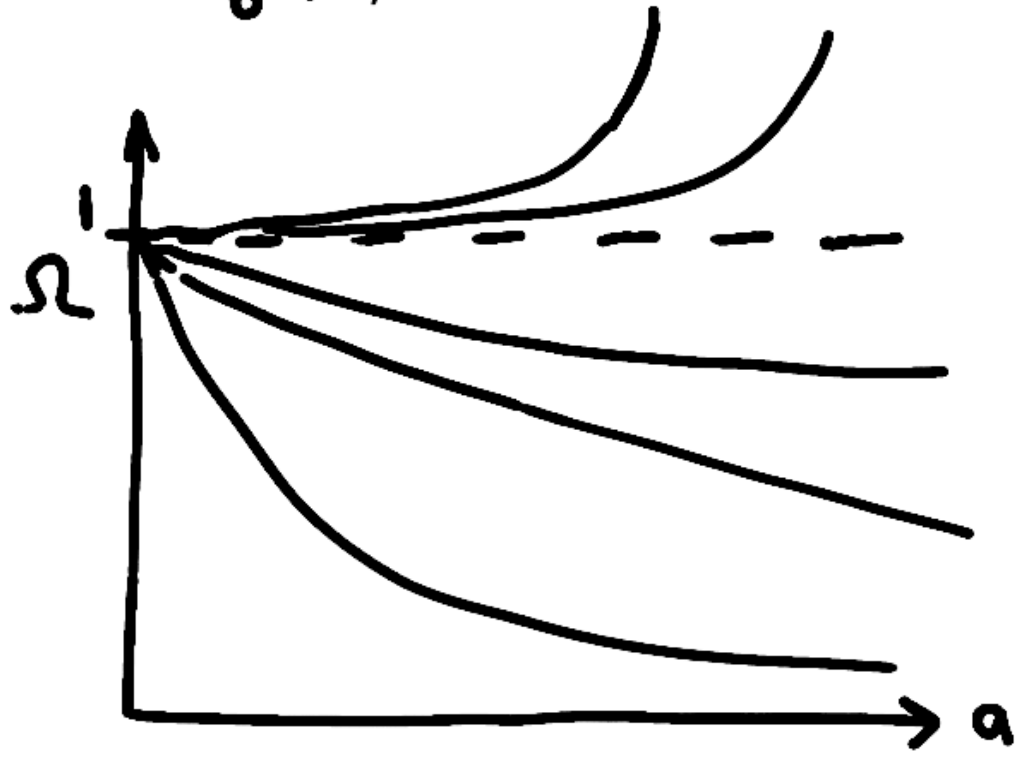
We can find how the contribution of matter to the total density changes with time:

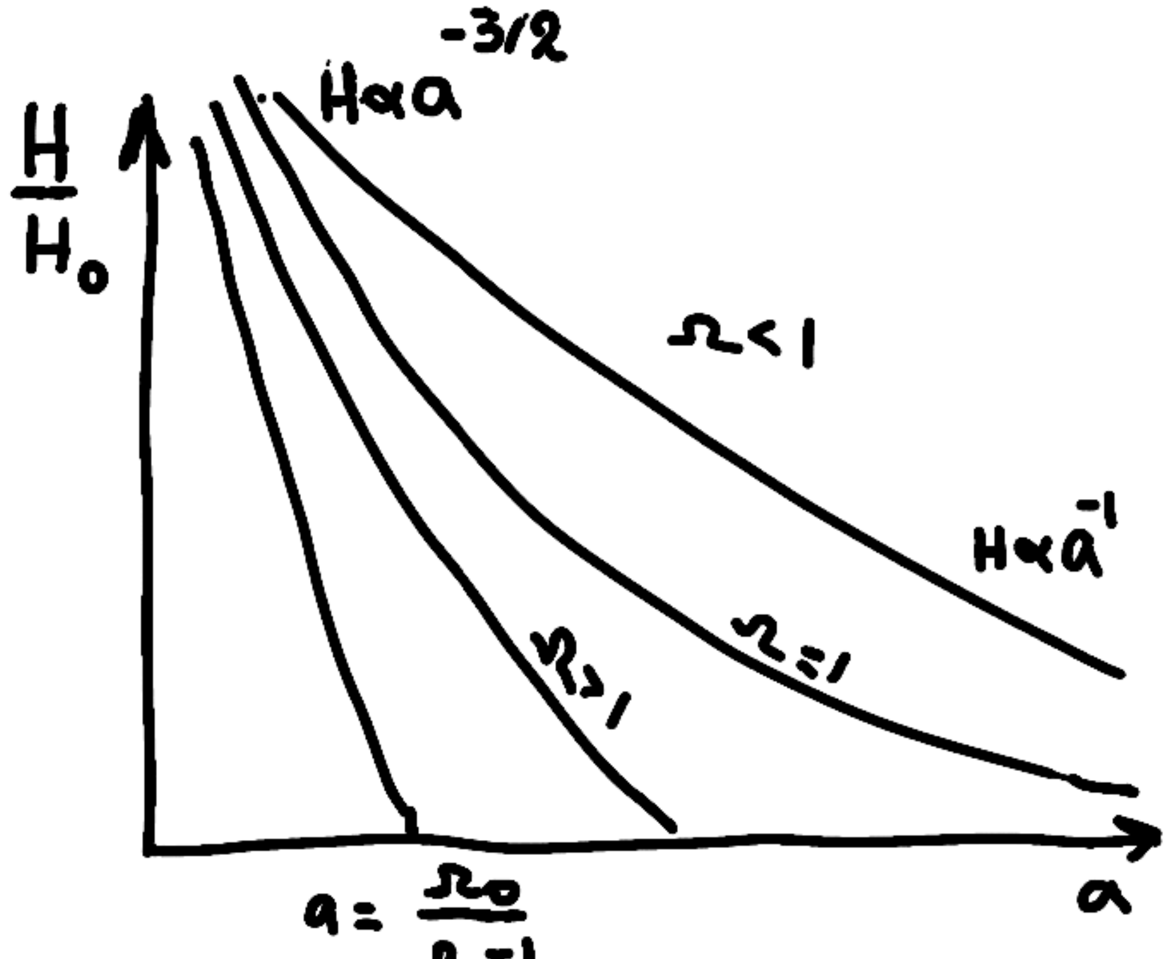
$$\Omega(t) \equiv \frac{\rho(t)}{\rho_{cr}(t)} = \frac{\Omega_0}{a^3}\left(\frac{H_0}{H}\right)^2 \Rightarrow$$

$$\Omega(t) = \frac{\Omega_0}{\Omega_0 + (1-\Omega_0)a}$$

If $\Omega_0 = 1$, then $\Omega(t) = 1$

If $\Omega_0 < 1$, then $\Omega(t) < 1$





$\Omega_0 = 1$

In this case $H^2 = H_0^2 / a^3 \Rightarrow \frac{1}{a}\frac{da}{dt} = \frac{H_0}{a^{3/2}}$

Solution of this equation is:

$$a(t) = \frac{3}{2}\left(H_0 t\right)^{2/3}$$

$$\Rightarrow \rho(t) = \frac{3 H_0^2}{8\pi G}\frac{1}{a^3} = \frac{1}{6\pi G t^2}$$

Age of the Universe $$t_u = \frac{2}{3}\frac{1}{H_0}$$

1.1 Definitions

It is convenient to express densities by introducing Ω - the contribution of particular component to the critical density of the Universe:

$$\Omega(t) \equiv \frac{\rho(t)}{\rho_{\rm cr}}, \qquad \rho_{\rm cr} = \frac{3H^2(t)}{8\pi G}. \tag{1}$$

We will use indexes to specify particular component and moment of time. For example, for the total mass of dark matter particles and baryons at $z = 0$ we use notation $\Omega_{m,0}$. For the Hubble constant at $z = 0$ we use notation H_0. Depending on the context, we often drop subscript "0" for omegas defined at redshift zero.

1.2 Nonrelativistic particles

The total number of nonrelativistic particles (e.g., at low redshifts electrons, protons and neutrons) is preserved in a sufficiently large comoving volume. Particles may participate in nuclear reactions that change abundances of different elements. Example can be the nuclear reactions inside stars that burn protons into He^4. However, reactions preserve the number of baryons. The same is for electrons. Elements can get ionized or can recombine changing the number of free electrons, but the total number of electrons – bound or not – is preserved. The fact that the mass of nonrelativistic particles is preserved in a comoving volume V means that the product of density by the volume is constant: $\rho_{\rm nr}V = const$. Because the comoving volume scales as $V \propto a^3$, where a is the expansion parameter, the density scales as $\rho_{\rm nr} \propto a^{-3}$. The normalization in this relation is a free parameter to be determined by observations.

If $\Omega_{m,0}$ is the contribution of the total mass of dark matter and baryons at $z = 0$, than we can write the density of the matter at any redshift as:

$$\rho_m(a) = \Omega_m(a)\rho_{\rm cr}(a) = \left(\frac{\Omega_{m,0}}{a^3}\right)\frac{3H_0^2}{8\pi G} \tag{2}$$

1.3 Relativistic particles

The situation is different for the relativistic particles. Their number-density and energy density are not preserved and is defined by the temperature of the Universe T and by the nature of the particles. For example, for photons the number density n_γ and the energy density $\rho_\gamma c^2$ are:

$$n_\gamma = \frac{2\zeta(3)}{\pi^2}\frac{(kT)^3}{(\hbar c)^3} = 414\left(\frac{T}{2.726K}\right)^3. \tag{3}$$

$$\rho_\gamma c^2 = \frac{2\pi^2 k^4}{30\hbar^3 c^3}T^4. \tag{4}$$

Here $\zeta(x)$ is the Riemann function: $\zeta(3) \approx 1.202$. The specific entropy of radiation is $s = (4/3)a_\gamma T^3$. Thus, the entropy of radiation and the number of photons are proportional one to the other.

Temperature of mass-less neutrinos is different from the temperature of photons. This is related with events that happened when the Universe was $\sim 1\,\mathrm{sec}$ old. At earlier stages the neutrinos were in nuclear equilibrium with other particles thanks to reactions such as $e + \bar{e} \leftrightarrow \nu_e + \bar{\nu}_e$. As the Universe was expanding, the density and temperature were decreasiong. At some moment the rate of those reactions became too small to keep neutrinos in equilibrium, and neutrinos became decoupled from the rest of the particles. That happened when the temperature of the Universe dropped to $T \sim 1$MeV. The electrons and positrons were still relativistic because their mass 0.5MeV is smaller than the temperature T. This means that the number and density of electron-positron pairs was extremely large - about twice larger than those of photons. As the Universe was further expanding and cooling the temperature dropped below 0.5MeV and electron-positron pairs annihilated $(e+\bar{e} \rightarrow \gamma+\gamma)$ depositing their tremendous energy into photons. As the result, the temperature of photons did not drop as much as it would have been otherwise. The final outcome of these events produced difference in temperatures of neutrinos and photons:

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma \qquad (5)$$

We thus expect that at present time the temperature of neutrinos should be $T_\nu = 0.71 \times 2.73K = 1.95K$.

After electron-positron annihilation there were no events that would modify the entropy (and, thus comoving number-density) of photons. The temperature was declining as $T \propto a^{-1}$. Because for relativistic particles $\rho \propto T^4$, the density of relativistic particles (photons + neutrinos) declines as $\rho_{rel} \propto a^{-4}$.

Following the same notations as for the non-relativistic particles, we can write the density of relativistic particles at $t > 1\,\mathrm{sec}$ as:

$$\rho_{\mathrm{rel}} = \left(\frac{\Omega_{\mathrm{rel},0}}{a^4}\right)\frac{3H_0^2}{8\pi G}. \tag{6}$$

Note, however, that we do know the normalization in this expressing because we know the temperature of CMB radiation.

1.4 Dark energy

We do not know what is the nature of the dark energy – the component, that dominates the expansion of the Universe at low redshifts. All observational results are consistent with the assumption the effective density of the dark energy does not change as the Universe expands. Thus we can write its contribution as:

$$\rho_\Lambda(a) = \Omega_{\Lambda,0}\frac{3H_0^2}{8\pi G}. \tag{7}$$

1.5 Friedmann equation

Combining these results we can finally write the Friedmann equation in the following form:

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \left(\frac{\Omega_{\mathrm{m},0}}{a^3} + \frac{\Omega_{\mathrm{rel},0}}{a^4} + \Omega_\Lambda\right). \tag{8}$$