Inflation

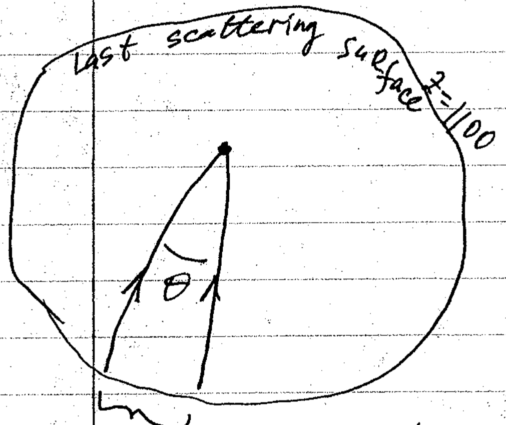
Problems of standard Hot Big Bang:

1) Large-scale homogeneity: why the Universe is homogeneous?

Universe is in causal contact within Hubble radius

$$d_H(t) = a(t)\int_0^t \frac{c\,dt}{a(t)} = \begin{cases} 3ct \propto a^{3/2}, & MD \\ 2ct \propto a^2, & RD \end{cases}$$

we see that at high redshift different areas of the Universe are all the same, but they have not been in causal contact yet. The best indicator is Cosmic Microwave Background (CMB)



Hubble radius at recombination

d_H at recombination is about 2°

$d_H = 3ct = 200h^{-1}$ Mpc

Entropy within the horizon is a measure, which is useful in this context. Entropy density $s \propto T^3$. Entropy per comoving volume is constant both on MD and RD stages.

$$S_H = s\, d_H^3 = \text{entropy in Hubble radius}$$
$$\simeq 10^{88} \text{ today (in some units)}$$

at recombination $S_H = 10^{83}$ (in the same units). Thus we are looking at $\sim 10^5$ independent volumes, which all have the same temperature

$k = H_0^2(1-\Omega_0)$

2) Spatial flatness / age of the Universe

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \Rightarrow \left|\Omega^{-1}(t) - 1\right| = \frac{3}{8\pi G}\frac{H_0^2}{\rho a^2} \propto \begin{cases} a, & MD \\ a^2, & RD \end{cases}$$

Thus

$$\left|\Omega^{-1} - 1\right| \lesssim \begin{matrix} 10^{-16} & \text{at BBN} \\ 10^{-60} & \text{at Planck time} \end{matrix}$$

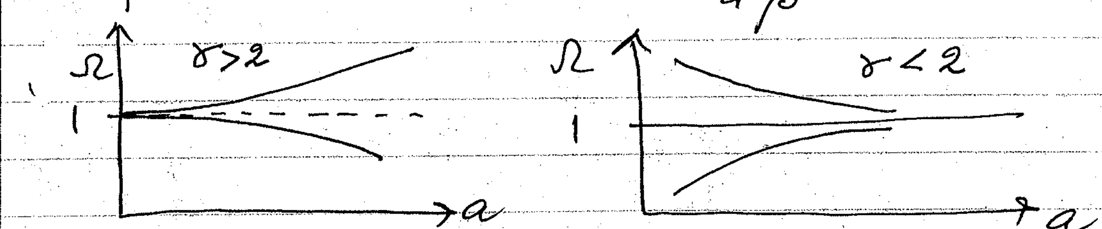
in order for $|\Omega^{-1} - 1| \sim 1$ today

Put this in other way: at Planck time we naively expect that fluctuation should on the order of unity. Then the life-time of the Universe (to collapse if $k=+1$ or to become curvature dominated if $k=-1$) is $\sim$ planck time $= 10^{-43}$ sec. The Universe is too old.

$$\left(\frac{\dot a}{a}\right)^2 - \frac{8\pi G}{3}\rho = -\frac{k}{a^2}$$

kinetic energy + potential energy = total energy

If $\rho \propto a^{-\gamma} \Rightarrow |\Omega^{-1} - 1| \propto \frac{1}{a^2\rho} \propto a^{\gamma-2}$ $\quad$ $\gamma = 3$, MD; $\gamma = 4$, RD



Big Bang $\qquad$ Inflation

$$\frac{d\rho}{dt} + 3\frac{\dot a}{a}(p+\rho) = 0 \qquad (c \equiv 1)$$

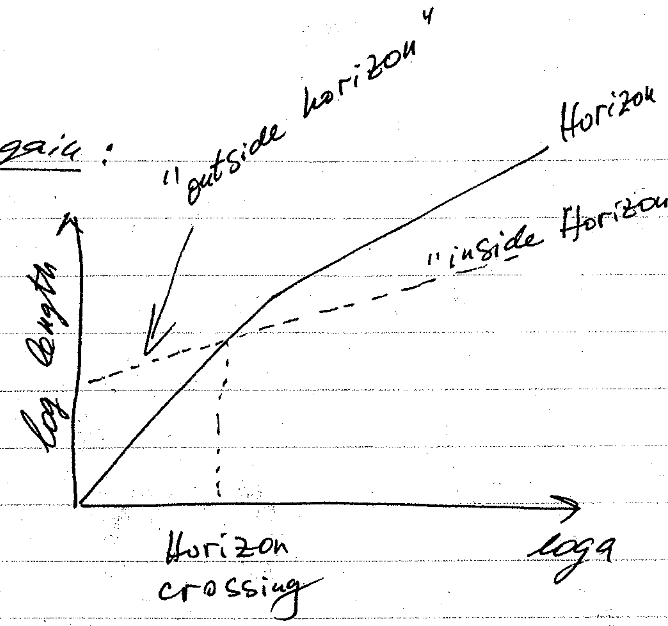
$$\Rightarrow \rho \propto a^{-3(1+w)} \Rightarrow w < -1/3 \Rightarrow \boxed{\rho + 3p < 0}$$

where $p = w\rho$

Large-scale homogeneity again:

$$d_H \propto \begin{cases} a^2, & RD \\ a^{3/2}, & MD \end{cases}$$

$$\lambda(t) = \lambda_0 a(t)$$



Big Bang

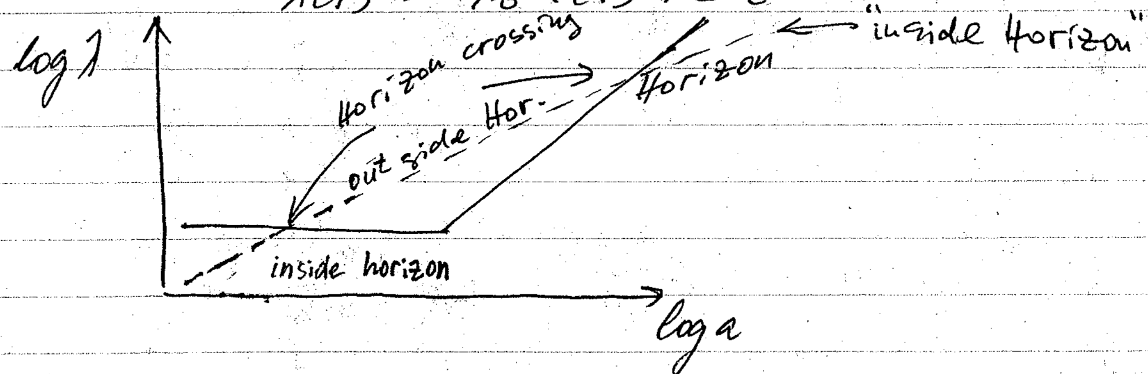
Inflation: if $\rho + 3p \le 0 \Rightarrow \ddot{a} \ge 0 \Rightarrow$

$\Rightarrow$ expansion is exponential $a \propto e^{Ht}$, where $H \sim$ constant

in this case

$$d_H = a(t) \int \frac{c\,dt}{a(t)} \sim \text{const}$$

$$\lambda(t) = \lambda_0 a(t) \sim e^{Ht}$$



$R_{Hubble} \sim ct \sim \lg a$

$\lambda \sim a$

idea: $H^2 = \frac{8\pi}{3} G\rho$; $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$

$c=1$ $\dot{\rho} + 3H(\rho + p) = 0$

if $p = -\rho$ (e.g. for cosmological constant)

$$\Downarrow$$

$$\rho = \text{const}$$

$$\Downarrow$$

$$H = \text{const}$$

$$\Downarrow$$

$$\frac{1}{a}\frac{da}{dt} = H \Rightarrow a \propto e^{Ht}.$$

Note that once inflation starts $\Rightarrow$

density of particles $\propto a^{-3} \to 0$
temperature $\propto a^{-1} \to 0$

So on inflation stage the Universe <u>empty</u> and <u>cold</u>

Once upon a time ....
long, long ago ....

The Vacuum Ruled

In the Beginning there was <u>Vacuum</u>

$\Rightarrow$ Start with a messy universe:
inhomogeneous + anisotropic + curved
mix all fields.
If in some region potential energy of vacuum is high
$\Rightarrow$ region expands exponentially
$\Rightarrow$ its volume increases relative to non-inflating regions

Inflation: main idea

(1) Fridmann equation: $H^2 = \frac{8\pi}{3} G\rho$

(2) acceleration of the Universe: $\frac{\ddot{a}}{a} = -\frac{4\pi}{3} G\left(\rho + \frac{3P}{c^2}\right)$

Here density ρ includes all possible components.

(3) Continuity equation: $\dot{\rho} + 3H\left(\rho + \frac{P}{c^2}\right) = 0$

Let's assume that equation of state is $P = -\rho c^2$.
In this case eq(3) gives $\dot{\rho} = 0 \Rightarrow \rho = \text{constant}$
The Friedmann eq(1) gives $H = \text{const}$ if $\rho = \text{const}$
From eq(2) we get that $\ddot{a} > 0 \Rightarrow$ universe accelerates!

For $H = \text{const}$ we get $\frac{1}{a}\frac{da}{dt} = H \Rightarrow a \propto e^{Ht}$

This is inflation: exponential growth of all scales.
We know that density of non-relativistic particles declines as $\rho_{non\text{-}rel} \propto a^{-3} \propto e^{-3Ht}$
Temperature of radiation goes as $T \propto a^{-1} \to 0$
Curvature $\sim a^{-2} \to 0$
So, once inflation starts, there are no other fields, particles in the Universe and it is cold!
And it is flat (or nearly so)

The 'only' thing we need is a field (or particles) with the equation of state $P = -\rho c^2$

Scalar fields are candidates for 'inflaton'. We assume that a scalar φ has potential energy $V(\varphi)$. We need to know how to write the energy-stress tensor for the field φ.

Lagrangian of the scalar field φ has two parts: kinetic energy and potential energy:

$$L_\varphi = \frac{1}{2}\left(g^{ij}\frac{\partial\varphi}{\partial x^i}\frac{\partial\varphi}{\partial x^j} - V(\varphi)\right) \qquad (*)$$

We are interested in fields, which (almost) have no dependency on coordinates. Thus, of all $\frac{\partial\varphi}{\partial x^i}$ derivatives in eq(*) the only terms, which are important are $\frac{\partial\varphi}{\partial t}$ and the first term is just $\sim\dot{\varphi}^2 \rightarrow$ equivalent of kinetic energy

an example of the second term is $V(\varphi) = \frac{m^2\varphi^2}{2}$, where m is mass of particle associated with field φ.

(**) Action of scalar field is written as: $S = \int d^4x \sqrt{-g}\, L_\varphi$. This gives energy-stress tensor:

$$T_{ij} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{ij}} = \frac{\partial\varphi}{\partial x^i}\frac{\partial\varphi}{\partial x^j} - g_{ij}L_\varphi$$

In locally Minkowski coordinates (equivalence principle!) the energy-stress tensor gives energy density (T_{00} component) and effective pressure (T_{ii} components)

$$\rho c^2 = T_{00} = \frac{1}{2}\left[\dot{\varphi}^2 + (\nabla\varphi)^2 + V(\varphi)\right]$$

$$P = T_{11} = \left(\frac{\partial\varphi}{\partial x^1}\right)^2 + \frac{1}{2}\left[\dot{\varphi}^2 - \frac{1}{3}(\nabla\varphi)^2 - V(\varphi)\right]$$

Terms with spacial gradients are expected to be small: $(\nabla\varphi)^2 \propto \left(\frac{\partial \cdots}{\partial r}\right)^2 \propto \frac{1}{a^2} \to 0$

If the scalar field φ is dominated by potential energy (thus, if $\dot{\varphi}^2$ is small as compared with $V(\varphi)$):

$$P = -\rho c^2$$

This is what we need for inflation.

Eq(**) for action also gives an equation for evolution of the scalar field (called Klein-Gordon):

$$\ddot{\varphi} + 3H\dot{\varphi} + V'(\varphi) = 0, \quad \text{where } V'(\varphi) = \frac{\partial V}{\partial\varphi}$$

This is the relativistic analog of the wave equation.

For simple case $V(\varphi) = \frac{1}{2}m^2\varphi^2$ we get:

$$\left(g^{ij}\frac{\partial\varphi}{\partial x^i}\right)_{;j} + m^2\varphi = 0 \Rightarrow (\Box + m^2)\varphi = 0$$

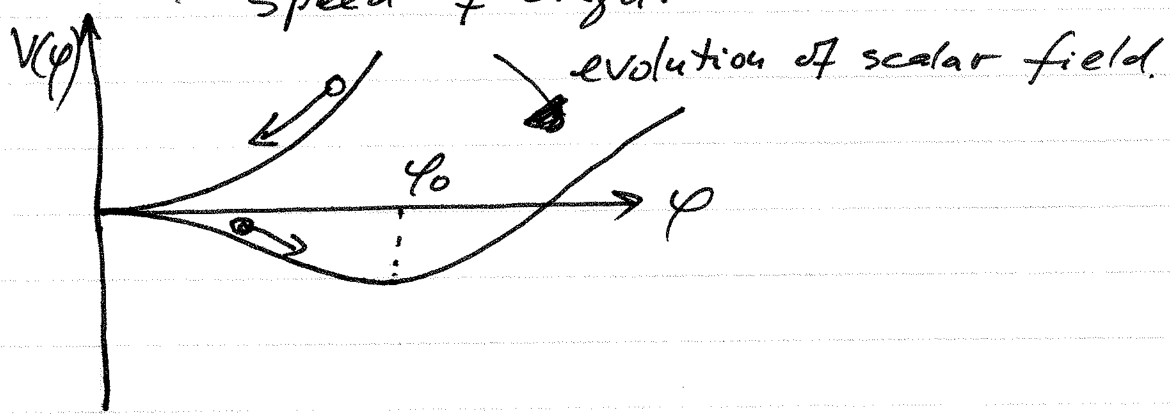
$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} + \nabla^2\varphi + m^2\varphi = 0$$

without expansion it gives $\varphi \propto e^{i\omega t}$

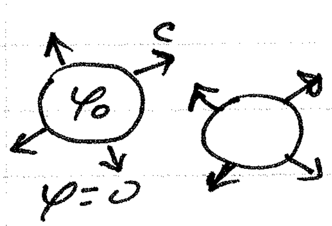
History: different models

Guth scenario

1) initial state has high temperature
2) as universe expands and cools, it gets into meta-stable state of faulse vaccuum
This is when the inflation takes place
3) Inflation ends with a phase transition into true vaccuum with some value φ_0.
This transition goes through formation of bubles of true vaccuum. Collision of bubles reheats the universe. Bubles expand with the speed of linght



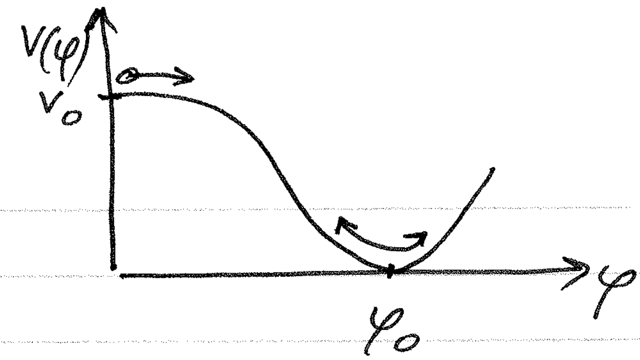
Problem: - collisions of bubles produce too large fluctuations



- inflation never ends because false vacuum expands exponentially - faster than velocity of light

Advantage: solved flatness and horizon problems

5 "New inflationary scenario"



1) Initially field φ starts with high energy $V_0 = V(0)$
2) It slowly rolls down producing inflation
3) Once the field gets close to φ_0 it oscillates. Coupling to other (normal) fields results in reheating and decaying of the field φ itself

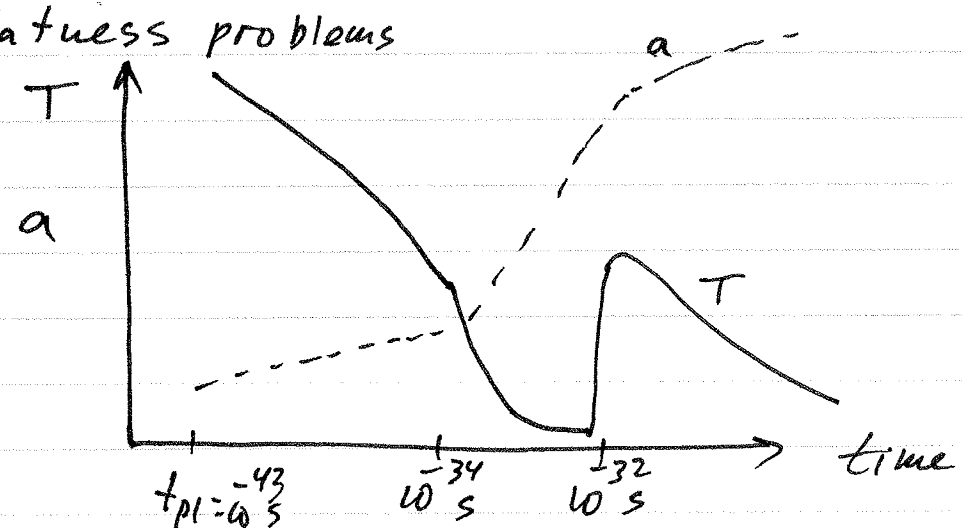
Problems: (1) amplitude of fluctuations must be $\sim 10^{-4} \sim 10^{-5}$, which can be acheived only if $V(\varphi)$ is very flat at $\varphi \simeq 0$. For example, for

$$V(\varphi) = V_0 - \frac{\lambda}{4}\varphi^4 \text{ we need}$$

very small $\lambda \simeq 10^{-12} - 10^{-14}$

Why it should so small when one expects $\lambda \sim 1$

(2) In order to produce small fluctuations inflation should start at low values of $V(\varphi)$. Temperature of Universe is also $T^4 \simeq V(0)$ is low. Because we expect that Universe starts at $t_{planck} \simeq 10^{-43}$ s, new inflationary scenario waits too long before inflation starts. This does not solve age and flatness problems

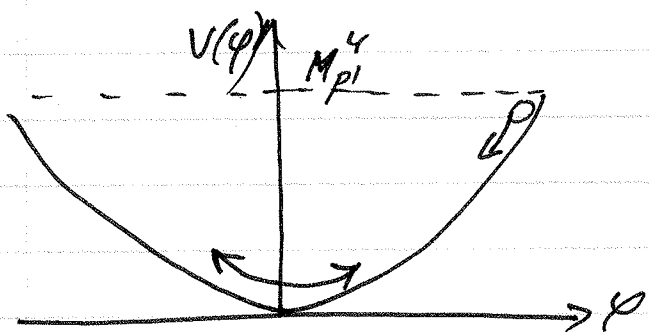


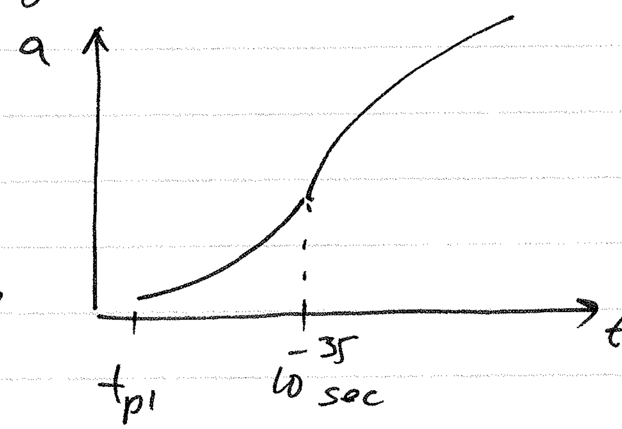
6 Chaotic inflation

→ Universe starts at ~ Planck energy ~ M_{pl}^4

→ Initially φ slowly rolls down and experiences large quantum-mechanical fluctuations

→ Once $V(\varphi)$ gets small and $\dot{\varphi}^2$ is big, field φ oscillates and decays into normal fields ending inflation and reheating Universe





Initial large quantum jumps produce phenomenon called eternal inflation

Dynamics of the scalar field and Slow-roll approximation

It is customary to change units in such a way that $\hbar = c = G = 1$. This brings in Planck mass into our equations because we still need to keep dimensions (and scales) right.

In those units, time has units of $[t] = M_{pl}^{-1}$
Length $[l] = M_{pl}^{-1}$
density $[\rho] = [\text{Energy}] = M_{pl}^{4}$

mnemonics: trace dependance on G of $M_{pl}, t_{pl}, \ldots$ and substitute G by M_{pl}^{-2}

Thus, our Friedmann equation takes form

$$H^2 = \frac{8\pi}{3}\frac{\rho}{M_{pl}^2} - \frac{k}{a^2} \qquad (1)$$

We know that for a scalar field the density ρ_φ is

$$\rho_\varphi = \frac{\dot{\varphi}^2}{2} + V(\varphi) \qquad (2)$$

// here I neglected $\frac{(\nabla\varphi)^2}{2}$ term

Now we need to add an equation for $\varphi(t)$
That was Klein-Gordon equation, which, in expanding coordinates, takes form

$$\ddot{\varphi} + 3H\dot{\varphi} + V'(\varphi) + \Gamma\dot{\varphi} = 0 \qquad (3)$$

where $V'(\varphi) = \frac{dV}{d\varphi}$ and Γ is a constant, which "couples" φ with other fields and eventually leads to the reheating of the Universe

Γ is small for the stages we are dealing with now

Given the potential $V(\varphi)$, it is not difficult to solve equations (1-3)

"Slow roll"

At the beginning of inflation H is large and the term with $\ddot{\varphi}$ is small and can be neglected. We also neglect $\Gamma\dot{\varphi}$ (because $\Gamma \ll 3H$)

(*) Thus $\boxed{3H\dot{\varphi} = -V'(\varphi)}$ ⇐ slow-roll approximation

This must be valid for some time for inflation to occur.

Find conditions on $V(\varphi)$ for slow-roll approximation

Density should be dominated by the potential: $\rho \approx V(\varphi)$, which means that $\frac{\dot{\varphi}^2}{2} \ll V(\varphi)$ (**)

$\Rightarrow H^2 \simeq \frac{8\pi}{3}\frac{V(\varphi)}{m_{pe}^2}$. Use this with (*) and (**) to exclude $\dot{\varphi}$:

$$\boxed{\frac{V'}{V}\cdot m_{pe} \ll (48\pi)^{1/2}}$$

Another condition - on the curvature of $V(\varphi)$ - follows from the condition $\ddot{\varphi} \ll 3H\dot{\varphi}$

Because the Hubble constant changes slowly, we can neglect its derivative with time. Then we can write: (from $3H\dot{\varphi} \simeq V'$)

$$\ddot{\varphi} \simeq \frac{(V')^{\cdot}}{3H} = \frac{1}{3H}\cdot\frac{d}{dt}(V'(\varphi)) = \frac{1}{3H}\frac{dV'}{d\varphi}\cdot\frac{d\varphi}{dt} = \frac{V''\dot{\varphi}}{3H}$$

This gives the second condition for slow-rolling

$$\boxed{V'' \ll 9H^2 \approx \frac{24\pi\, V(\varphi)}{m_{pe}^2}}$$

Now we can estimate how much the Universe expands on inflation. The number of e-folds of growth in the scale factor that occurs as φ rolls from φ_1 to φ_2 is $\quad (a \propto e^{\int H dt})$

$$N_e \equiv \ln\left(\frac{a_2}{a_1}\right) \equiv \int_{t_1}^{t_2} H dt = -\frac{8\pi}{m_{pe}^2} \int_{\varphi_1}^{\varphi_2} \frac{V(\varphi)}{V'(\varphi)} d\varphi$$

Here we used $dt = d\varphi/\dot{\varphi}$; $H^2 = \frac{8\pi}{3}\frac{V}{m_{pe}^2}$; $\dot{\varphi} = -\frac{V'}{3H}$

If we assume that $\varphi_{final} \simeq 0$ and $\varphi_{initial} \simeq m_{pe}$
Then the first condition for slow-roll $\left(\frac{V}{V'}\frac{1}{m_{pe}} \gg \frac{1}{(48\pi)^{1/2}}\right)$
gives a low limit on the number of e-folds

$$N_e \gg \left(\frac{8\pi}{6}\right)^{1/2} \simeq 2$$

For inflation to be successful, we need to blow 1 cm to the whole Universe. This corresponds to

$$N_e > 62$$

So, while the slow-roll approximation tells us that we will probably have many e-folds, estimates must be done carefully.

<u>Example</u>: Chaotic inflation with $V(\varphi) = \frac{m^2\varphi^2}{2}$

$\ddot{\varphi} + 3H\dot{\varphi} = -m^2\varphi \qquad \frac{m}{m_{pe}} \simeq 10^{-7}$ for $\frac{\delta\rho}{\rho} \sim 10^{-5}$

slow-roll: $3H\dot{\varphi} = -m^2\varphi$; $H^2 = \frac{4\pi}{3}\frac{m^2\varphi^2}{m_{pe}^2}$

This gives $H(\varphi) = \sqrt{\frac{4\pi}{3}}\frac{m\varphi}{m_{pe}} \Rightarrow \dot{\varphi} = -\frac{m\, m_{pe}}{2\sqrt{3\pi}} \Rightarrow$

$\Rightarrow \varphi = \varphi_0 - \frac{m\, m_{pe}}{2\sqrt{3\pi}} \cdot t$ || Now we need $t \to a$

$$H = \frac{1}{a}\frac{da}{dt} \Rightarrow \frac{da}{a} = \sqrt{\frac{4\pi}{3}}\frac{m}{m_{pe}}\varphi dt$$

$$\varphi dt = \frac{\varphi d\varphi}{\dot\varphi} = -\frac{2\sqrt{3\pi}}{m\, m_{pl}} \cdot d\left(\frac{\varphi^2}{2}\right) \Rightarrow \frac{da}{a} = -\frac{2\pi}{m_{pl}^2} d\varphi^2$$

thus the solution is

$$a = a_0 \exp\left[\frac{2\pi}{m_{pl}^2}\left(\varphi_0^2 - \varphi^2(t)\right)\right]$$

If we assume that inflation starts when $V_{init} \simeq M_{pl}^4$ then

$$V_{init} = \frac{m^2 \varphi_0^2}{2} \simeq M_{pl}^4 \Rightarrow \varphi_0 \simeq \frac{M_{pl}^2}{m}$$

if inflation ends with $\varphi \ll \varphi_0$, we have the number of e-folds for this model

$$N_e = \ln\left(\frac{a}{a_0}\right) \simeq 2\pi\left(\frac{m_{pl}}{m}\right)^2 \simeq 10^{14}$$

Fluctuations

- because of the quantum nature, always there are small quantum fluctuations in the field φ. On average those fluctuations are $= 0$ (so, $\langle\delta\varphi\rangle = 0$) But $\langle(\delta\varphi)^2\rangle$ might not be zero.
- Usually those fluctuations are so small, that we can neglect them. Usually even statistical fluctuations ($\sim 1/N^{1/2}$ due to finite number of particles) are small too.
- But now the Universe expands very fast. If a quantum "fluke" can get stretched beyond the horizon <u>before</u> it disappears, it survives because waves longer than horizon do not have enough time to oscillate. Those waves will result in fluctuations of the scalar field $\Rightarrow$ the fluctuations in φ result in different (local) expansion rate $\Rightarrow$ fluctuations in metric $\Rightarrow$ fluctuations in density.

Results from the quantum theory:

$$|\delta\varphi| \simeq \frac{H}{2\pi}$$

For simplicity we assume that $V(\varphi) = \frac{m^2\varphi^2}{2}$ (chaotic inflation)

Thus, quantum fluctuations produce $\sqrt{|\delta\varphi|^2} = \frac{m\varphi}{\sqrt{3\pi}\, M_{pl}}$ in $\Delta t = 1/H$

At the same time (so in $\Delta t = 1/H$) the field will roll down by

$$\Delta\varphi = \frac{m M_{pl}}{2\sqrt{3\pi}} \cdot \frac{1}{H} \qquad // \text{remember, that here } \dot{\varphi} = -\frac{m M_{pl}}{2\sqrt{3\pi}}$$

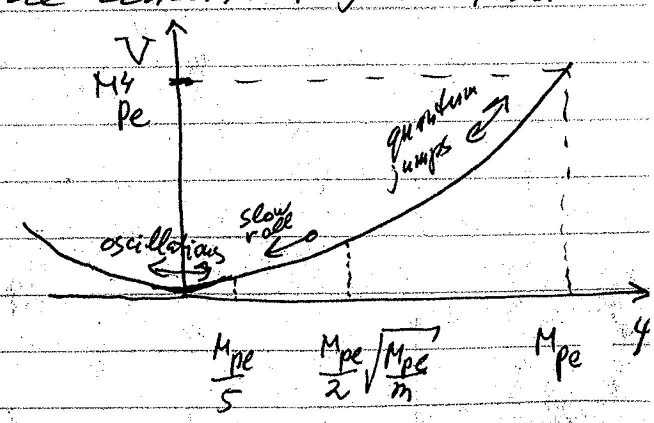
on this stage (slow-roll) $H \simeq \sqrt{\frac{4\pi}{3}} \cdot \frac{m \cdot \varphi}{M_{pl}}$

Thus $\Delta\varphi = \frac{M_{pl}^2}{4\pi\varphi}$

Now, we compare $\underbrace{\Delta\varphi}_{\text{roll-down}}$ and $\underbrace{\delta\varphi}_{\text{quantum fluctuation}}$

If $\delta\varphi$ is bigger than $\Delta\varphi$, the field φ actually (almost) does not roll down. It is "randomly" "walking" up and down. The condition for that is

$$\varphi > \frac{1}{2} M_{pl} \sqrt{\frac{M_{pl}}{m}}$$



When writing equation for the evolution of φ with time:

$$\ddot{\varphi} + 3H\dot{\varphi} = -V'_{\varphi} = -m^2\varphi$$

we neglected terms related with spacial gradients. It should be $(\Box + m^2)\varphi = 0$
Now those terms are important because they have information of the waves we are interested in. Thus, the equation for waves $\delta\varphi = \delta\varphi(t, \vec{x})$ is

$$\delta\ddot{\varphi} + 3H\delta\dot{\varphi} - \underbrace{\frac{1}{a^2}\nabla^2(\delta\varphi)}_{\text{spacial gradients}} = -m^2\delta\varphi \qquad (*)$$

$\Delta r \sim a \sim \Delta x$, $r = at$

Equation $(*)$ has two types of solution:

1) for $k \gg H \gg m^2$ it describes oscilations of $\delta\varphi$
$\left(\delta\ddot{\varphi} = \frac{1}{a^2}\nabla^2(\delta\varphi)\right)$

2) for $k \ll H \Rightarrow$ frozen disturbance due to the friction term $3H\delta\dot{\varphi}$
In this case the solution is just
$\delta\varphi = \text{const}$

Since the potential energy depends on the field φ, fluctuations in φ result in fluctuations in the energy density: $\delta\rho_\varphi = \frac{\partial V}{\partial\varphi}\cdot\delta\varphi \simeq \frac{V}{\varphi}\delta\varphi$

$$\Rightarrow \frac{\delta\rho_\varphi}{\rho_\varphi} \simeq \frac{\delta\varphi}{\varphi}$$

Use fourier spectrum: $\delta\varphi_k = \int d^3x \exp(i\vec{k}\vec{x})\,\delta\varphi(\vec{x})$
The amplitude of fluctuations from quantum theory is

$$\Delta\varphi_k^2 \equiv \underset{\text{volume}}{k^3}\,\underset{\text{amplitude}}{|\delta\varphi_k|^2} = \left(\frac{H}{2\pi}\right)^2$$

In this expression the right-hand-side does <u>not</u> depend on wave-number k. It tells us that for each typical scale $\lambda = \frac{2\pi}{k}$ the <u>amplitude</u> of fluctuations is the same: $\boxed{\delta\varphi_k^2 \propto k^{-3}}$

We need to express these results in our usual quantitaties.
The fluctuations in the total energy density ρ_φ result in fluctuations of metric, which in our language are fluctuations of the gravitational potential. The fluctuations in the density are related with fluctuations in the potential (grav. potential) via the Poisson equation:

$$\Delta\varphi = 4\pi G\rho$$

$\leftarrow$ not the energy density, density of matter at later stages

$\uparrow$ grav. potential, not scalar field

For waves with wave-number k, this gives

$$k^2\varphi_k \sim \delta\rho_k$$

$$\Rightarrow \boxed{\left(\frac{\delta\rho}{\rho}\right)_k^2 \propto k^1}$$