Zeldovich Approximation

If initial fluctuations are smooth, we can estimate the moment of collapse and the whole dynamics of the collapse without assuming the spherical symmetry or any kind of homogeneity of initial density perturbations as we did for the spherical infall model or for the homogeneous ellipsoid model.

In the Zeldovich approximation we follow the motion of a fluid element. This is done using the Lagrangian approach to the equations of hydrodynamics.

$\vec{r}$ = proper coordinate, changes with time

$\vec{x}$ = comoving coordinate, changes with time

$\vec{q}$ = lagrangian coordinate, does not depend on time

If we know comoving coordinates, we always can find proper coordinates using :

$$\boxed{\vec{r} = a(t)\,\vec{x}(t)}$$

$\vec{q}$ = lagrangian coordinate is a label of particles. This is comoving coordinate of particle in the absence of perturbations

According to Zeldovich (1970) the relation between $\vec{x}(t)$ and $\vec{q}$ is given by:

$$\boxed{\vec{x}(t) = \vec{q} + b(t)\,\vec{S}(\vec{q})}$$

Here $b(t)$ = growth rate of fluctuations in the linear regime (= $a(t)$ for a flat CDM universe)

$\vec{S}(\vec{q})$ is a displacement vector. Note that it does <u>not</u> depend on time

Find peculiar velocity predicted by ZA: assume that $b = a \quad \Omega = 1$

$$\vec{v}(t) = a\frac{d\vec{x}}{dt} = a \cdot \dot{a}(t)\vec{S}(\vec{q}) = a^2 H \vec{S}(\vec{q})$$

$$\vec{v}(t) \parallel \vec{S}(\vec{q})$$

Find peculiar gravitational acceleration

$$\vec{g} = \frac{d\vec{v}}{dt} + \frac{\dot{a}}{a}\vec{v} = a\ddot{x} + 2\frac{\dot{a}}{a} a\dot{x} = a\ddot{a}\vec{S} + 2\dot{a}\dot{a}\vec{S} =$$

$$= a^2\left(\frac{\ddot{a}}{a} + 2H^2\right)\vec{S} =$$

$$= \frac{3}{2}H^2 a^2 \vec{S} = \frac{3}{2}H\vec{v} \Rightarrow \vec{g} \parallel \vec{v}$$

So, according to ZA particles are moving along straight lines.

From the linear theory we know that perturbations do not have rotation. This can be written as

$$\mathrm{curl}\,\vec{v} = 0 \Rightarrow \mathrm{curl}\,\vec{S} = 0$$

Thus, the displacement vector is a gradient of a scalar function:

$$\vec{S}(\vec{q}) = \vec{\nabla}\Phi(\vec{q})$$

In order to find the relation between $\vec{S}, \Phi$ and the density perturbation $\frac{\delta\rho}{\rho}$

we use the continuity equation in lagrangian formulation of equateons of hydrodynamics

$$\boxed{\rho(x,t)\,d^3x = \rho_b\,d^3q}$$

Now we need to find the Jacobian of transformation from lagrangian to comoving coordinates:

$$\rho(x,t) = \frac{\rho_b}{\det\left(\frac{\partial^3 x_i}{\partial^3 q_j}\right)}, \quad \text{where} \quad \frac{\partial^3 x_i}{\partial q_j} = \begin{pmatrix} \frac{\partial x_1}{\partial q_1} & \frac{\partial x_2}{\partial q_1} & \frac{\partial x_3}{\partial q_1} \\ \frac{\partial x_1}{\partial q_2} & \dots & \\ \frac{\partial x_1}{\partial q_3} & \dots & \end{pmatrix}$$

$$\frac{\partial x_i}{\partial q_j} = \delta_{ij} + a\frac{\partial s_i}{\partial q_j} = \delta_{ij} + a\frac{\partial \varphi}{\partial q_i \partial q_j} \Rightarrow \frac{\partial x_i}{\partial q_j} = \text{symmetric tensor} \qquad \frac{\partial x_i}{\partial q_j} = \frac{\partial x_j}{\partial q_i}$$

$a\frac{\partial^2 \Phi}{\partial q_i \partial q_j}$ is called "deformation tensor"

Use the linear theory to normalize ZA: $a\frac{\partial^2 \Phi}{\partial q_i \partial q_j} \ll 1$

$$\det\left(\frac{\partial^3 x_i}{\partial^3 q_j}\right) \simeq \left(1 + a\frac{\partial^2 \Phi}{\partial q_1^2}\right)\left(1 + a\frac{\partial^2 \Phi}{\partial q_2^2}\right)\left(1 + a\frac{\partial^2 \Phi}{\partial q_3^2}\right) \simeq 1 + a\left(\frac{\partial^2 \Phi}{\partial q_1^2} + \frac{\partial^2 \Phi}{\partial q_2^2} + \frac{\partial^2 \Phi}{\partial q_3^2}\right) = 1 + a\nabla^2 \Phi$$

Use these expressions to find the density in the linear regime:

$$\rho(x,t) \simeq \frac{\rho_b}{1 + a\nabla^2 \Phi} \simeq \rho_b(1 - a\nabla^2 \Phi) \qquad \boxed{\frac{\delta\rho}{\rho} = -a\nabla^2 \Phi}$$

$$\bar{\Phi} = \frac{1}{(2\pi)^3}\int d^3k\, \bar{\Phi}_k\, e^{-i\vec{k}\vec{q}} \Rightarrow \boxed{\delta_k = a k^2 \bar{\Phi}_k} \Rightarrow \varphi = -\frac{3}{2} a^3 H^2 \bar{\Phi}$$

φ: grav. potential

$\bar{\Phi}$: velocity potential

This is used to –) normalize the Zeldovich approximation

–) to set initial conditions for numerical simulations

Logic: $\delta_k^2 = A k^n T^2(k,t) \Rightarrow \bar{\Phi}_k \Rightarrow \frac{\delta_k}{(ak^2)} \Rightarrow S_k \Rightarrow$

$\Rightarrow \vec{x}(\vec{q},t)$

<u>Nonlinear regime</u>: $\frac{\partial x_i}{\partial q_j}$ is a symmetric tensor.

By changing a system of coordinates, any symmetric tensor can be made diagonal: There is a system of coordinates for which

$$\frac{\partial x_i}{\partial q_j} = \begin{pmatrix} \frac{\partial x_1}{\partial q_1} & & 0 \\ & \frac{\partial x_2}{\partial q_2} & \\ 0 & & \frac{\partial x_3}{\partial q_3} \end{pmatrix}$$

$\frac{\partial x_i}{\partial q_i}$ = eigenvalues of the tensor

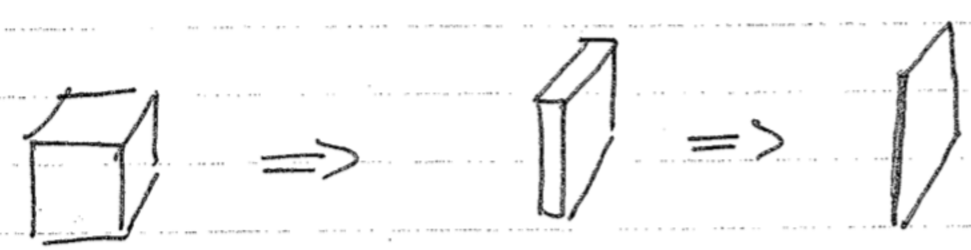
$\lambda_i \equiv$ eigenvalues of deformation tensor $\frac{\partial^2 \Phi}{\partial q_i \partial q_j}$

$$\det\left(\frac{\partial x_i}{\partial q_j}\right) = (1 - a\lambda_1(q))(1 - a\lambda_2(q))(1 - a\lambda_3(q))$$

Using these expressions for the deformation tensor we can finally write the density:

$$\rho(x,t) = \frac{\rho_0}{(1-a\lambda_1)(1-a\lambda_2)(1-a\lambda_3)}$$

if $\lambda_1 \geq \lambda_2 \geq \lambda_3$, at time t defined by condition $a(t)\lambda(q) = 1$, the density becomes (formally) infinite



The Zeldovich approximation is exact for 1-d fluctuation before the crossing of trajectories.

1D case: $x = q + a(t)\sin(kq)$

$v = a \cdot \dot{a} \sin(kq)$

$\frac{dx}{dq} = 1 + a k \cos(kq)$

$\rho = \frac{\rho_b}{1 + k a(t) \cos(kq)}$