Spherical Infall Model

1 MOTIVATION.

This is a simplified model of evolution of fluctuations in the nonlinear regime. Its goal is to find whether a fluctuation is going to collapse. If it is going to collapse, what would be its maximum radius and at what time it is going to finally collapse. Having these predictions the model further makes another step and estimates the density and radius of the virialized object.

In reality the model does not predict the correct dynamics of the collapsed object. For example, the basic assumption of the spherical infall model that as the region collapses it maintains its nearly spherical shape is just wrong. However, in spite or maybe because of its simplicity, the model captures basic trends of the collapse and has numerous applications. It is widely used for statistics such as the halo mass function or halo virial radius.

There are some variations of this model. We use the simplest one by considering a universe dominated by the cold dark matter with no vacuum energy. We assume that at some initial moment of time t_i we consider a spherical perturbation with radius R_i and constant overdensity:

$$\rho(R) = \rho_b(1+\delta_i), \quad R < R_i, \tag{1}$$

where the background density is

$$\rho_b = \frac{3H^2(t)}{8\pi G}\Omega(t). \tag{2}$$

We also assume that the region initially expands with the rest of the Universe and does not have peculiar velocity: $v_{\rm pec} = 0$. This is not correct for the growing mode of perturbations where density perturbation is accompanied by velocity perturbation. We will correct for this effect later when we normalize the growth of perturbations in the spherical model to the predictions of thelinear theory. For now we assume there are initial peculiar velocity is zero.

2 MAXIMUM RADIUS OF EXPANSION.

To find the maximum radius of expansion we consider a particle at the radius of the sphere. Initial kinetic energy per unit mass of the particle is $K_i = H_i^2 R_i^2/2$ and the initial potential energy per unit mass is $W_i = -GM/R_i$, where M is the mass of the sphere. We write:

$$W_i = -\frac{4\pi}{3}\rho_{b,i}(1+\delta_i)R_i^2 = \frac{H_i^2 R_i^2}{2}\Omega_i(1+\delta_i). \tag{3}$$

Here the initial density of the background are:

$$\rho_{b,i} = \frac{3H_i^2}{8\pi G}\Omega_i. \tag{4}$$

Thus, the total energy of the particle is

$$E_i = K_i + W_i = -\frac{H_i^2 R_i^2}{2}\Omega_i\left(1+\delta_i - \frac{1}{\Omega_i}\right). \tag{5}$$

Because the energy of the particle is preserved we now need to find the energy of the particle at the moment of maximum expansion and equate it to the initial energy. At the moment of maximum expansion the velocity of the sphere is zero, which gives us $K_{\rm max} = 0$. The potential energy is

$$W_{\rm max} = -\frac{GM}{R_{\rm max}} = -\frac{H_i^2 R_i^3}{2R_{\rm max}}\Omega_i\,(1+\delta_i) \tag{6}$$

Use the conservation of energy: $E_i = W_{\rm max}$ to find the radius:

$$\frac{Rmax}{R_i} = \frac{1+\delta_i}{1+\delta_i - 1/\Omega_i}. \tag{7}$$

For the flat universe or for the model at high redshift when $\Omega_i \approx 1$ we find:

$$\frac{Rmax}{R_i} = \frac{1+\delta_i}{\delta_i} \approx \frac{1}{\delta_i}. \tag{8}$$

For example, a sphere with initial small overdensity 1/100 will expand 100 times before it totally breaks from the Hubble flow and stops expanding.

Only fluctuations that have total energy negative will stop expansion at some moment. This defines a critial level of overdensity:

$$1+\delta_i - 1/\Omega_i \geqslant 0 \qquad \Rightarrow \quad \delta_{cr} = \frac{1}{\Omega_i} - 1. \tag{9}$$

For the flat Universe $\delta_{cr} = 0$.

(II) Expansion law of a spherical perturbation

$E < 0$, $\Omega_b = 1$

$$\boxed{\frac{1}{2}\left(\frac{dR}{dt}\right)^2 = \frac{GM}{R} + E}$$

Growing solution of this equation can be written in parametric form:

$$\boxed{R = A(1-\cos\theta)}, \quad \boxed{t = B(\theta - \sin\theta)}$$

Now we need to find A and B in terms of δ_i, R_i, t_i

For $\theta \ll 1$ $\cos\theta \simeq 1 - \frac{\theta^2}{2}$, $\sin\theta \simeq \theta - \frac{\theta^3}{6}$

$$1-\cos\theta \approx \frac{\theta^2}{2}; \quad \theta - \sin\theta \approx \frac{\theta^3}{6}$$

Thus,

$$\left.\begin{aligned} R &= A\frac{\theta^2}{2} \\ t &= \theta^3 \cdot \frac{B}{6} \end{aligned}\right] \Rightarrow \underline{R = \frac{A}{2}\frac{6^{2/3}}{B^{2/3}} t^{2/3}} \quad (R \propto t^{2/3})$$

for small θ we can neglect E in the energy equation: $\frac{1}{2}\left(\frac{dR}{dt}\right)^2 = \frac{GM}{R} + 0$

$$\rightarrow GM = \frac{R}{2}\left(\frac{dR}{dt}\right)^2$$

$$\frac{dR}{dt} \simeq \frac{2}{3}\frac{R}{t} \rightarrow \boxed{A^3 = GM\,B^2} \rightarrow$$

$$\rightarrow R \approx \left(\sqrt{\frac{9}{2}GM} \cdot t\right)^{2/3}$$

$\Rightarrow$ solution to the second-order terms in θ gives

$$R \approx \left(\sqrt{\tfrac{9}{2}GM}\, t\right)^{2/3}\left[1 - \frac{1}{20}\left(\frac{6t}{B}\right)^{2/3}\right]$$

this gives density: $\rho = \dfrac{3M}{4\pi R^3} = \dfrac{1}{6\pi G t^2}\left(1 + \dfrac{3}{20}\left(\dfrac{6t}{B}\right)^{2/3}\right)$

for $t = t_i$ $\quad \rho = \rho_{cr}(1+\delta_i) \Rightarrow$

$$B = \frac{6t_i}{\left(\frac{20}{3}\delta_i\right)^{3/2}} \Rightarrow A = \left(GMB^2\right)^{1/3} = \frac{3}{10}\frac{R_i}{\delta_i}$$

Thus the final solution of the evolution of expanding sphere is

$$\begin{cases} R = \dfrac{3}{10}\dfrac{R_i}{\delta_i}(1-\cos\theta) \\[2ex] t = \dfrac{6t_i}{\left(\frac{20}{3}\delta_i\right)^{3/2}}(\theta - \sin\theta) \end{cases}$$

this gives overdensity: $\dfrac{\rho}{\rho_b} = \dfrac{9}{2}\dfrac{(\theta - \sin\theta)^2}{(1-\cos\theta)^3}$

Maximum of expansion is at $\theta = \pi$

$$\frac{R_{max}}{R_i} = \frac{1}{\frac{5}{3}\delta_i}$$

|| when we assume that $v_{pec,i} = 0 \Rightarrow$ this is not all in the growing mode. Only $\frac{3}{5}$ of initial fluctuation is growing

$$\frac{t_{max}}{t_i} = \frac{6\pi}{\left(\frac{20}{3}\right)^{3/2}} \cdot \frac{1}{\delta_{i,gr}^{3/2}}$$

$\leftarrow$ here δ_i is the amplitude of the growing mode

$$\delta_{i,gr} = \frac{3}{5}\delta_{i,tot}$$

Density at the turn-around moment is

$$\left.\frac{\rho}{\rho_b}\right|_{\text{max expansion}} = \frac{9}{16}\pi^2 = 5.55 \quad \text{or} \quad \frac{\delta\rho}{\rho} \simeq 4.55$$

if we would extrapolate the linear growth to the max. expansion,

$$\left.\frac{\delta\rho}{\rho}\right|_{\text{linear}} = (6\pi)^{2/3}\,\frac{3}{20} \simeq 1.06$$

Moment of collapse

for $\theta = 2\pi$ $R = 0 \rightarrow$ collapse of the fluctuation

Note that $\boxed{t_{collapse} = 2\, t_{max}}$

After the moment of collapse the system quickly (2-3 dynamical times) approaches the virial equilibrium with

$$E \simeq -\frac{W_{virial}}{2}$$

conservation of energy:

$$E_{max} = E_{virial} \Rightarrow$$

$$\Rightarrow W_{max} = \frac{W_{virial}}{2} \Rightarrow$$

$$\Rightarrow \boxed{R_{virial} \simeq \frac{R_{max}}{2}}$$

at the moment of collapse,

$$\left.\frac{\rho}{\rho_b}\right|_{collapse} = \left(\frac{\rho}{\rho_b}\right)_{max} \cdot \left(\frac{R_{max}}{R_{virial}}\right)^3 \cdot \left(\frac{t_{collapse}}{t_{max}}\right)^2 =$$

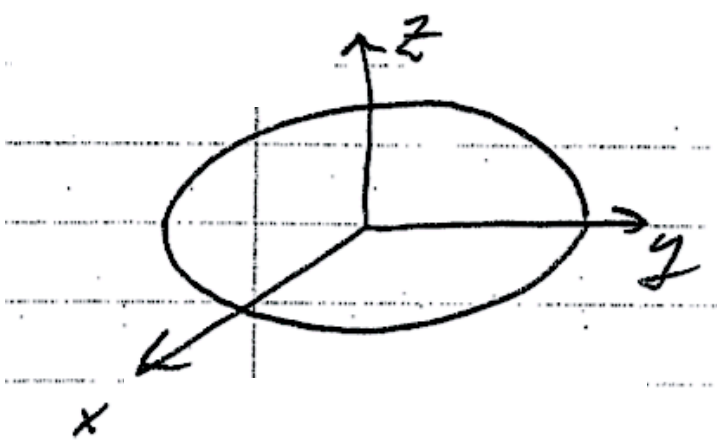
$$= \left(\frac{\rho}{\rho_b}\right)_{max} \cdot 2^3 \cdot 4 = 18\pi^2 = 178$$

linear extrapolation gives

$$\left.\frac{\delta\rho}{\rho}\right|_{linear,\ collapse} = 1.68$$

Homogeneous ellipsoid model

This is a modification of the top-hat model. It assumes that we deal with a small initial flattening. How will it grow as the fluctuation expands, turns around and collapses?



ellipsoid: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

grav. potential of a homogeneous ellipsoid is a quadratic function of coordinates:

$$\Phi = \pi G\rho\left(Ax^2+By^2+Cz^2\right)$$

The Poisson equation gives $A+B+C=2$ $(\nabla^2\Phi=4\pi G\rho)$

For a sphere $A=B=C \rightarrow A=2/3$

For an oblate spheroid $(a=b>c)$

$$A = B = \frac{(1-e^2)^{1/2}}{e^2}\left[\frac{\arcsin e}{e} - (1-e^2)^{1/2}\right]$$

$$C = \frac{2(1-e^2)^{1/2}}{e^2}\left[\frac{1}{(1-e^2)^{1/2}} - \frac{\arcsin e}{e}\right]$$

$$e = \left(1 - \frac{c^2}{a^2}\right)^{1/2} = \text{eccentricity}$$

If the spheroid does not rotate, the equations of expansion of semiaxes are

$$\begin{cases} \dfrac{1}{a}\dfrac{d^2 a}{dt^2} = -2\pi G \rho A(e) \\[2ex] \dfrac{1}{c}\dfrac{d^2 c}{dt^2} = -2\pi G \rho C(e) \end{cases} \Rightarrow \text{plots}$$