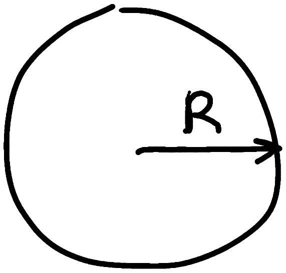
Friedmann Equation: Newtonian derivation

Consider a sphere, which expands in a homogeneous Universe. For non-relativistic particles the mass inside the sphere is constant. We need to find how the radius of the sphere changes with time. Later we will add corrections due to effects of GR.



$$\frac{d^2R}{dt^2} = -\frac{GM(R)}{R^2} = -\frac{4\pi}{3}G\rho R \qquad (*)$$

Use the fact that mass inside comoving radius is preserved:

$$M = \frac{4\pi}{3}\rho R^3 \rightarrow R^3\frac{d\rho}{dt} + 3\rho R^2\frac{dR}{dt} = 0 \Rightarrow$$

$$\frac{d\rho}{dt} + 3H\rho = 0$$

Multiply eq(*) by $2\dot{R}$ we get $2\dot{R}\ddot{R} = -\frac{8\pi G}{3}\rho R\dot{R}$

The l.h.s. is a full derivative: $(\dot{R}^2)^{\cdot}$

The r.h.s. also can be written as a full derivative: $\rho R\dot{R} = -\rho R^3\frac{d}{dt}\left(\frac{1}{R}\right)$

Now, we can integrate the equation to get:

$$(\dot{R})^2 = \frac{8\pi G}{3}\rho R^2 + A, \qquad A = \text{const}$$

Divide both sides by R^2 and use $\frac{\dot{R}}{R} = \frac{\dot{a}}{a} = H$

$$(**)\quad H^2 = \frac{8\pi G}{3}\rho + \frac{B}{a^2}, \qquad \text{here } B = A\left(\frac{a}{R}\right)^2 = \text{Const}$$

Let's first find the constant B. At a=1, $H=H_0$, and introduce a new constant:

$$\rho_0 = \Omega_0\frac{3H_0^2}{8\pi G}$$

Put these relations into eq(**) and find that $B = H_0^2(1-\Omega_0)$

Thus, $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{H_0^2(1-\Omega_0)}{a^2} \quad \Big| \begin{array}{l} P=0 \\ \Lambda=0 \end{array}$

Because for non-relativistic particles mass in a comoving volume is preserved, we can re-write the Fieadmann equation in a different form:

$$\left(\frac{H}{H_o}\right)^2 = \frac{\Omega_o}{a^3} + \frac{1-\Omega_o}{a^2} \qquad \Big\| \; \rho = \frac{\rho_o}{a^3} = \frac{\Omega_o}{a^3}\frac{3H_o^2}{8\pi G}$$

Critical density: $$\rho_{cr}(t) \equiv \frac{3H^2}{8\pi G}$$

Special case: $\Omega_o = 1 \Rightarrow H \propto a^{-3/2}$

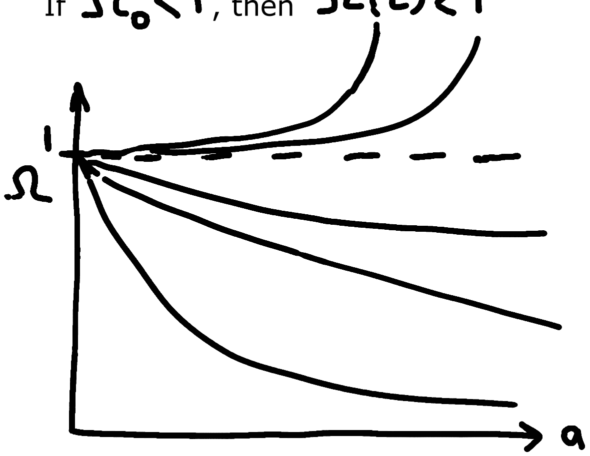
We can find how the contribution of matter to the total density changes with time:

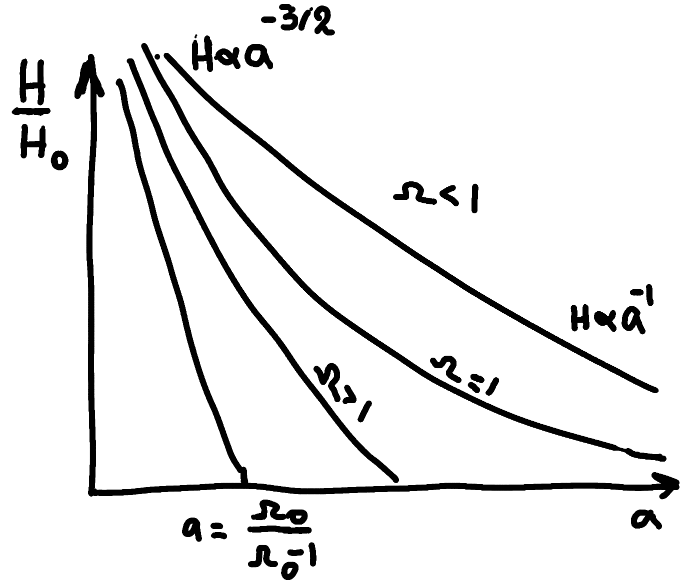
$$\Omega(t) \equiv \frac{\rho(t)}{\rho_{cr}(t)} = \frac{\Omega_o}{a^3}\left(\frac{H_o}{H}\right)^2 \Rightarrow$$

$$\Omega(t) = \frac{\Omega_o}{\Omega_o + (1-\Omega_o)a}$$

If $\Omega_o = 1$, then $\Omega(t) = 1$

If $\Omega_o < 1$, then $\Omega(t) < 1$





$\Omega_0 = 1$ In this case $H^2 = H_0^2/a^3 \Rightarrow \frac{1}{a}\frac{da}{dt} = \frac{H_0}{a^{3/2}}$

Solution of this equation is:

$$a(t) = \frac{3}{2}(H_0 t)^{2/3}$$

$$\Rightarrow \rho(t) = \frac{3H_0^2}{8\pi G}\frac{1}{a^3} = \frac{1}{6\pi G t^2}$$

Age of the Universe $t_U = \frac{2}{3}\frac{1}{H_0}$