Corrections due to General Relativity and relativistic effects

Logic: Use FRW metric, assume a simple form of energy-stress tensor, get Friedmann equation.

FRW: $ds^2 = c^2 dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2 d\varphi^2)\right]$

Einstein's equations: $R_{ij} - \frac{1}{2} g_{ij} R = \frac{8\pi G}{c^2} T_{ij}$

Ricci tensor: $R_{00} = -3\left(\frac{\dot{a}}{a}\right)^2$

$$R_{ij} = -\left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{2k}{a^2}\right] g_{ij}$$

Scalar curvature $R = -6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right]$

Energy-stress tensor: $T_{ij} = (\rho c^2 + P) u_i u_j - g_{ij} P + \Lambda g_{ij}$

$$\vec{u} = \frac{d\vec{x}}{ds} = \begin{cases} \frac{1}{\sqrt{1 - v^2/c^2}} \\ \frac{\vec{v}}{\sqrt{1 - v^2/c^2}} \end{cases}$$

The 0-0 component of Einstein's equations gives:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

The ii components give: $2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = -8\pi G p - \frac{k}{a^2}$

Combining those equations, we get:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right)$$

The 0-0 equation also can be written as: $\rho a^3 = \frac{3}{8\pi G} a(\dot{a}^2 + k)$
Differentiate it relative to time and use ii:

$$\frac{d}{dt}(\rho a^3) = \frac{3}{8\pi G}\left[\dot{a}(\dot{a}^2 + k) + 2a\dot{a}\ddot{a}\right] =$$

$$= \frac{3}{8\pi G} a^2 \dot{a}\, 8\pi G p = -p\frac{da^3}{dt}$$

This is the first law of thermodynamics: $dE = -p\,dV$
We can re-write it in the form of continuity equation

$$\frac{d\rho}{dt} + 3H\left(\rho + \frac{P}{c^2}\right) = 0$$

For non-relativistic particles P is much smaller than ρc^2. Thus, it can be neglected. This gives usual $\rho \propto a^{-3}$
For relativistic particles such as photons or neutrinos $P = \frac{\rho c^2}{3}$

There are more exotic particles and fields, which can give different relations between the pressure and density. Conventionally the equation of state is written in the form:

$$P = w \rho c^2$$

Parameter w is a very important for the evolution of the Universe:

$w = 0$ for non-relativistic particles

$w = 1/3$ Relativistic particles

$w = 1$ mass-less scalar fields

$w = -1/3$ Curvature. Cosmic strings (vacuum energy in 1d defects)

$w = -2/3$ Domain walls (vacuum energy trapped in 2d defects)

$w = -1$ Cosmological constant. Massive scalar fields

Case $K=0$, $P = \frac{\rho c^2}{3}$ Radiation dominated epoch.

In this case: $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) = -\frac{8\pi G \rho}{3}$

From continuity equation we get: $\rho = \rho_0 a^{-4}$

Substitute this into the Friedmann equation and integrate it:

$$a = \left(\frac{32}{3}\pi G \rho_0\right)^{1/4} t^{1/2} \Rightarrow \rho = \frac{3}{32\pi G t^2}$$