Zeldovich Approximation

Following text is taken from S.Borgani lecture: http://fisica.usac.edu.gt/public/curccaf_proc/borganihtml/node4.html

The Zeldovich approximation

A simple and elegant approximation to describe the non-linear stage of gravitational evolution has been developed by Zeldovich [38] (see the review by Shandarin & Zeldovich [28], for an exhaustive description of the Zeldovich approximation). In this approach, the initial matter distribution is considered to be homogeneous and collisionless. If the unperturbed (initial) Lagrangian coordinates of the particles are described by $\mathbf{q}$, then the Eulerian coordinates of the particles at the time t are given by

$$\mathbf{r}(\mathbf{q},t) \;=\; a(t)\,[\mathbf{q}+b(t)\,\mathbf{s}(\mathbf{q})]\,. \tag{15}$$

Here $a(t)$ is the cosmic expansion factor and $b(t)$ the growing rate of linear fluctuations, as provided by eq.(13). Moreover, the velocity term $\mathbf{s}(\mathbf{q})$, which provides the particle displacement with respect to the initial (Laplacian) position, is related to the potential $\Phi_o(\mathbf{q})$ originated by the initially linear fluctuations, according to

$$\mathbf{s}(\mathbf{q}) \;=\; \nabla\Phi_o(\mathbf{q})\,. \tag{16}$$

In order to better visualize the meaning of eq.(15), let us consider a pressureless and viscosity-free, homogeneous medium without any gravitational interaction. For this system, the Eulerian positions $\mathbf{x}$ of the particles at time t are related to the Lagrangian positions $\mathbf{q}$ by the linear relation

$$\mathbf{x}(\mathbf{q},t) \;=\; \mathbf{q}+\mathbf{v}(\mathbf{q})t\,, \tag{17}$$

being $\mathbf{v}(\mathbf{q})$ the initial velocity. The above expression is essentially analogous to the Zeldovich approximation (15), apart from the presence of the $a(t)$ term, which accounts for the background cosmic expansion, and of the $b(t)$ term, which accounts for the presence of gravity, giving a deceleration of particles along the trajectories (actually, $b(t)\propto t^{2/3}$ in a $\Omega=1$ matter dominated Universe).

Since at $t>0$ density inhomogeneities are created, mass conservation requires that $\rho(\mathbf{r},t)\,d\mathbf{r}=\rho_o\,d\mathbf{q}$, so that the density field as a function of Lagrangian coordinates reads

$$\rho(\mathbf{q},t) \;=\; \rho_o\left|\frac{\partial\mathbf{r}}{\partial\mathbf{q}}\right| \;=\; \frac{\bar\rho}{\left|\delta_{ij}-b(t)\,\frac{\partial s_i}{\partial q_j}\right|}\,. \tag{18}$$

Here the *deformation tensor* $\partial r_i / \partial q_j$ accounts for the gravitational evolution of the fluid, while $\bar{\rho} = (a_o/a)^3 \rho_o$ is the mean density at time t. At the linear stage, when $b(t)\,\mathbf{s}(\mathbf{q}) \ll 1$, eq.(18) can be approximated by

$$\rho(\mathbf{q},t) \;\simeq\; \bar{\rho}\,[1 - b(t)\nabla_q \cdot \mathbf{s}(\mathbf{q})]\,, \tag{19}$$

so that $\bar{\rho}\delta(\mathbf{x}) \simeq -b(t)\,\nabla_q \cdot \mathbf{s}(\mathbf{q})$ and we recover (the growing mode of) the linear solution.

More in general, since the expression (16) for $\mathbf{s}(\mathbf{q})$ makes the deformation tensor a real symmetric matrix, its eigenvectors define a set of three principal (orthogonal) axes. After diagonalization, eq.(18) can be written in terms of its eigenvalues $-\alpha(\mathbf{q})$, $-\beta(\mathbf{q})$ and $-\gamma(\mathbf{q})$, which give the contraction or expansion along the three principal axes:

$$\rho(\mathbf{q},t) \;=\; \frac{\bar{\rho}}{[1-b(t)\,\alpha(\mathbf{q})]\,[1-b(t)\,\beta(\mathbf{q})]\,[1-b(t)\,\gamma(\mathbf{q})]}\,. \tag{20}$$

If the eigenvalues are ordered in such a way that $\alpha(\mathbf{q}) \geq \beta(\mathbf{q}) \geq \gamma(\mathbf{q})$, then, as $b(t)$ grows, the first singularity in eq.(20) occurs in correspondence of the Lagrangian coordinate $\mathbf{q}_1$, where α attains its maximum positive value α_{max}, at the time t_1 such that $b(t_1) = \alpha_{max}^{-1}$. This corresponds to the formation of a pancake (sheet-like structure) by contraction along one of the principal axes. For this reason, Zeldovich [38] argued that pancakes are the first structures formed by gravitational clustering. Other structures like filaments and knots come from simultaneous contractions along two and three axes, respectively. Doroshkevich [8] evaluated the probability distribution for the three eigenvalues in the case of a Gaussian random field and concluded that simultaneous vanishing of more than one of them is quite unlikely. Thus, in this scenario, pancakes are the dominant features arising from the first stages of non-linear gravitational clustering.

The Zeldovich approximation predicts the first non-linear structure to arise in correspondence of the high peaks of the $\alpha(\mathbf{q})$ field and represents a significant step forward with respect to linear theory and in fact it has been successfully applied to describe the large scale clustering in the distribution of galaxy clusters.

However, within the Zeldovich prescription, after a pancake forms in correspondence of crossing of particle orbits, such particles continue travelling along straight lines, according to eq.(15). Viceversa, in the framework of a realistic description of gravitational dynamics, we expect that the potential wells, that correspond to non-linear structures, should be able to retain particles and to accrete from surrounding regions.

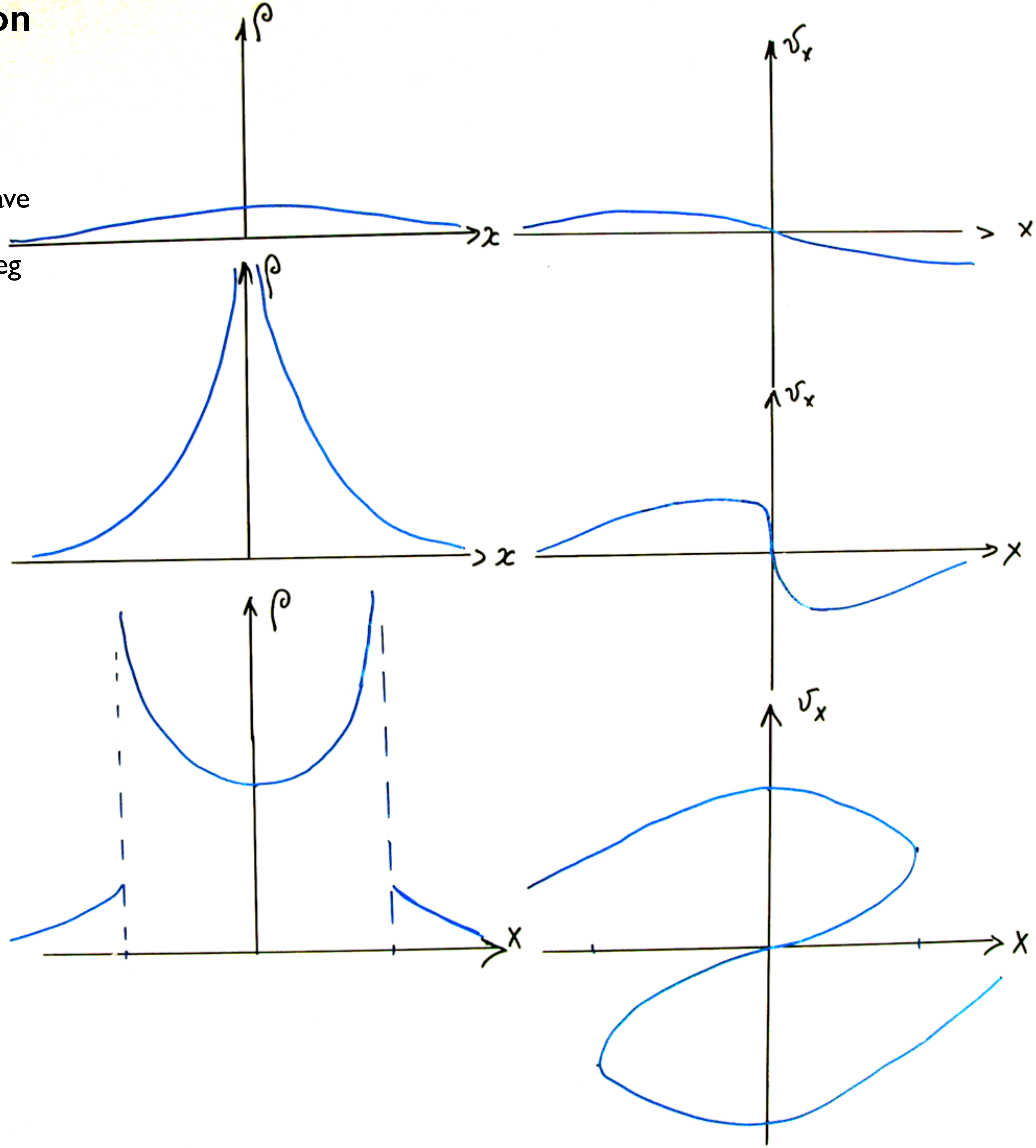
End of text is taken from S.Borgani lecture: http://fisica.usac.edu.gt/public/curccaf_proc/borganihtml/node4.html

Evolution of a 1D perturbation

Early stages of evolution: small-amplitude wave produces an overdensity around x= 0. Perturbation in velocity goes in phase (90 deg difference) with the density perturbation

Formation of caustic: density is infinite at x=0. The curve of V(x) has vertical slope at the same x=0 distance.

At later stages particles still keep moving along their initial direction: they do not turn around in the Zeldovich approximation. Now we have two caustics, which move away from the central x=0 plane. At each point inside the region of the caustics there are three flows.



1D case

$$x(q,t) = q + t \cdot v_0(q) \; .$$

$$\rho(q,t) = \frac{\rho_0}{1 + t \cdot \alpha(q)}$$

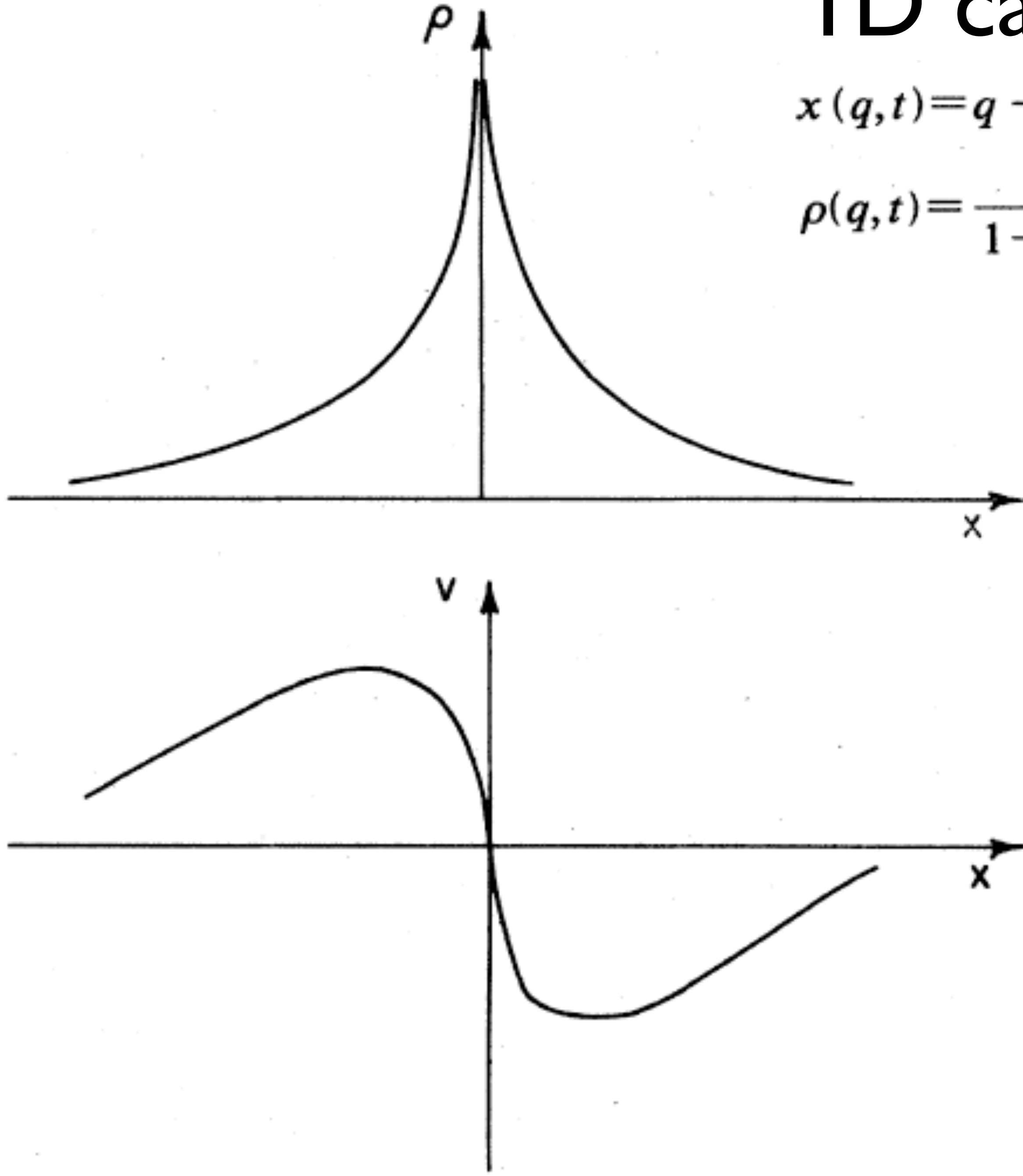


FIG. 2. The formation of a pancake begins with the development of a singularity in the density distribution. It has a particular form $\rho \propto |x|^{-2/3}$ in the vicinity of $x=0$ and is schematically illustrated in panel (a). Exactly at the time of the singularity formation, the velocity field develops a vertical tangent at the position of the singularity. This is schematically illustrated in panel (b). At $x=0$, $dv/dx=\infty$.

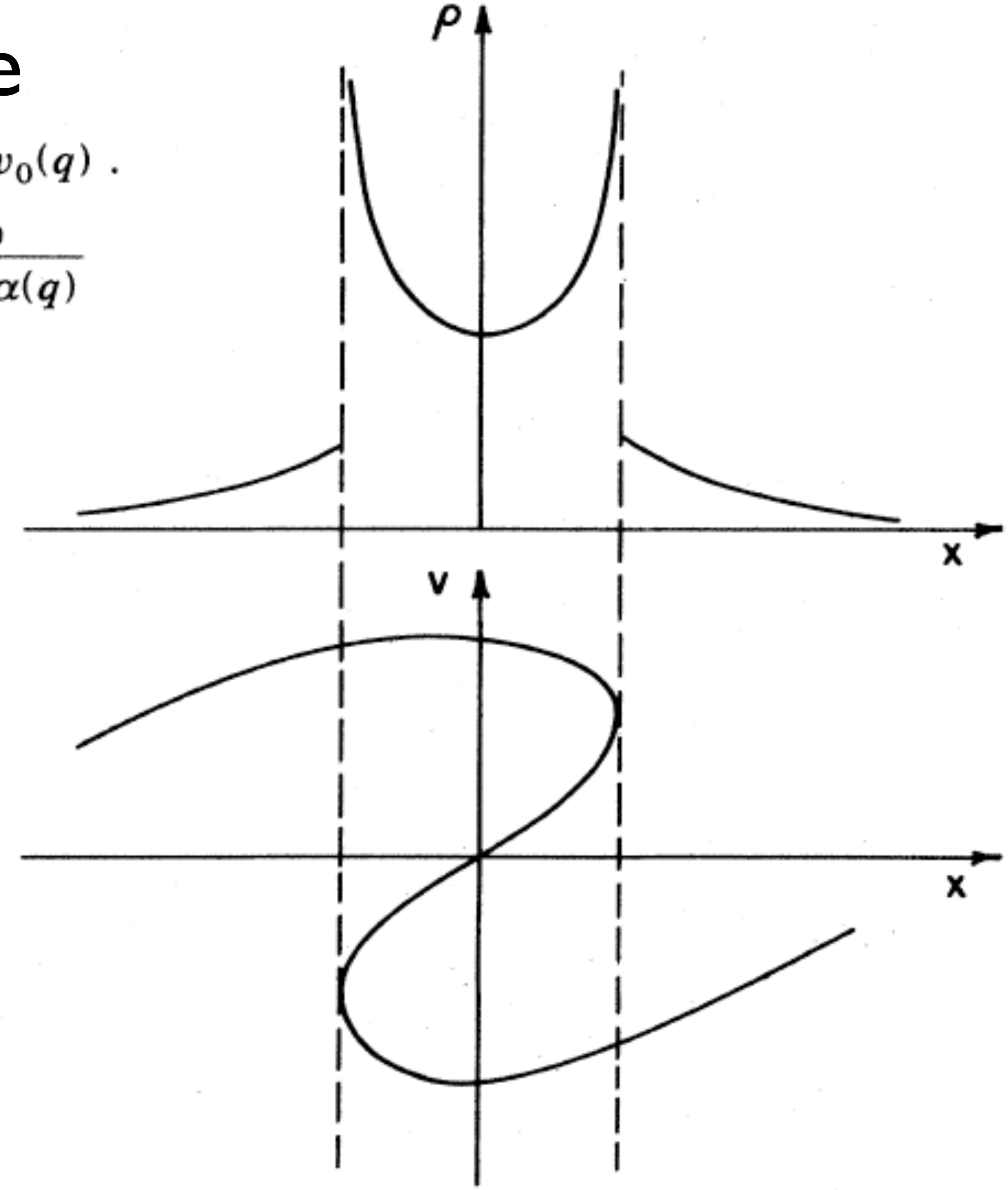
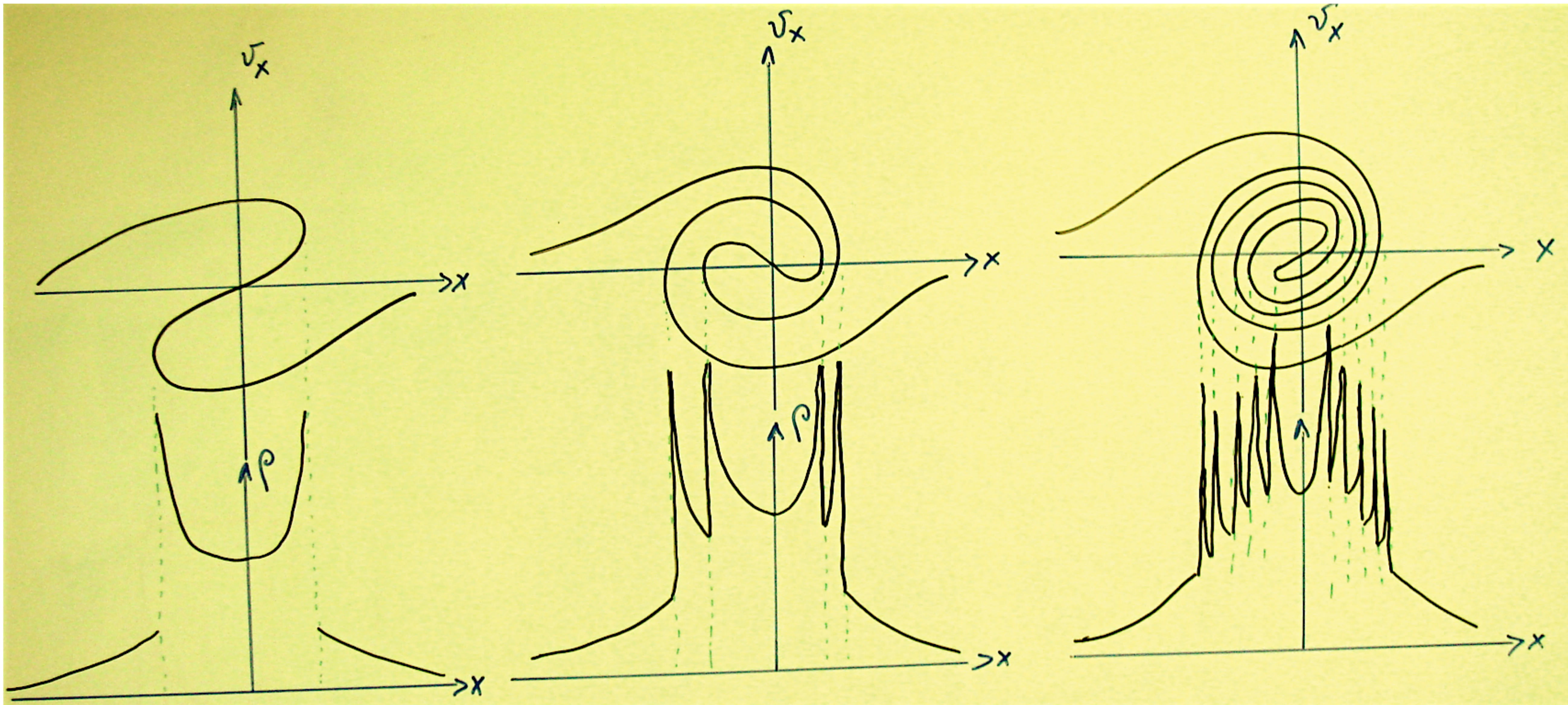


FIG. 3. Density and velocity distribution a short time after pancake formation. The boundaries of the three-stream flow region are indicated by dashed lines. Inside the three-stream region in the vicinity of the border $\rho \propto |x-x_s|^{-1/2}$.

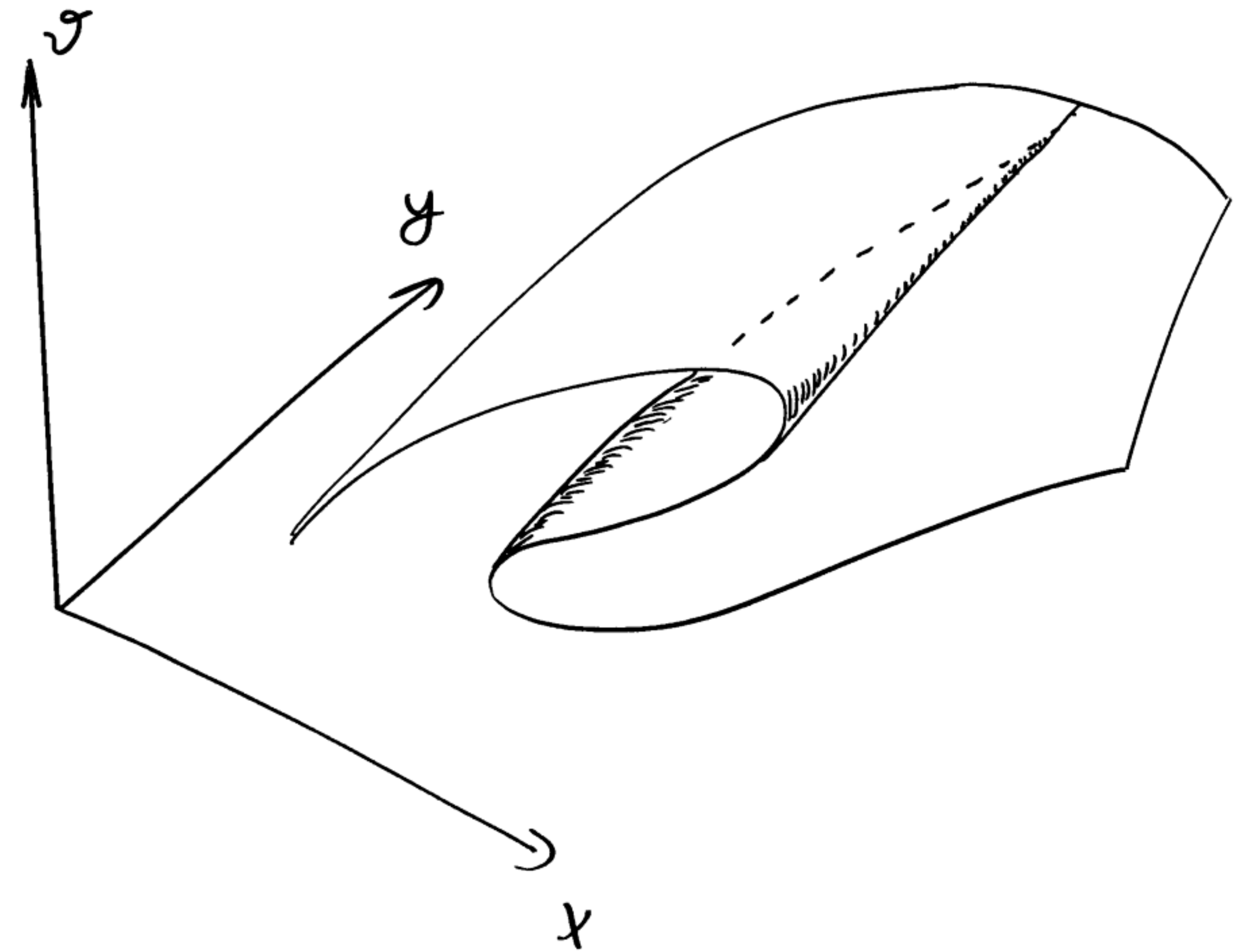
Shandarin & Zeldovich 1982

Evolution of a 1D perturbation: beyond Zeldovich approximation

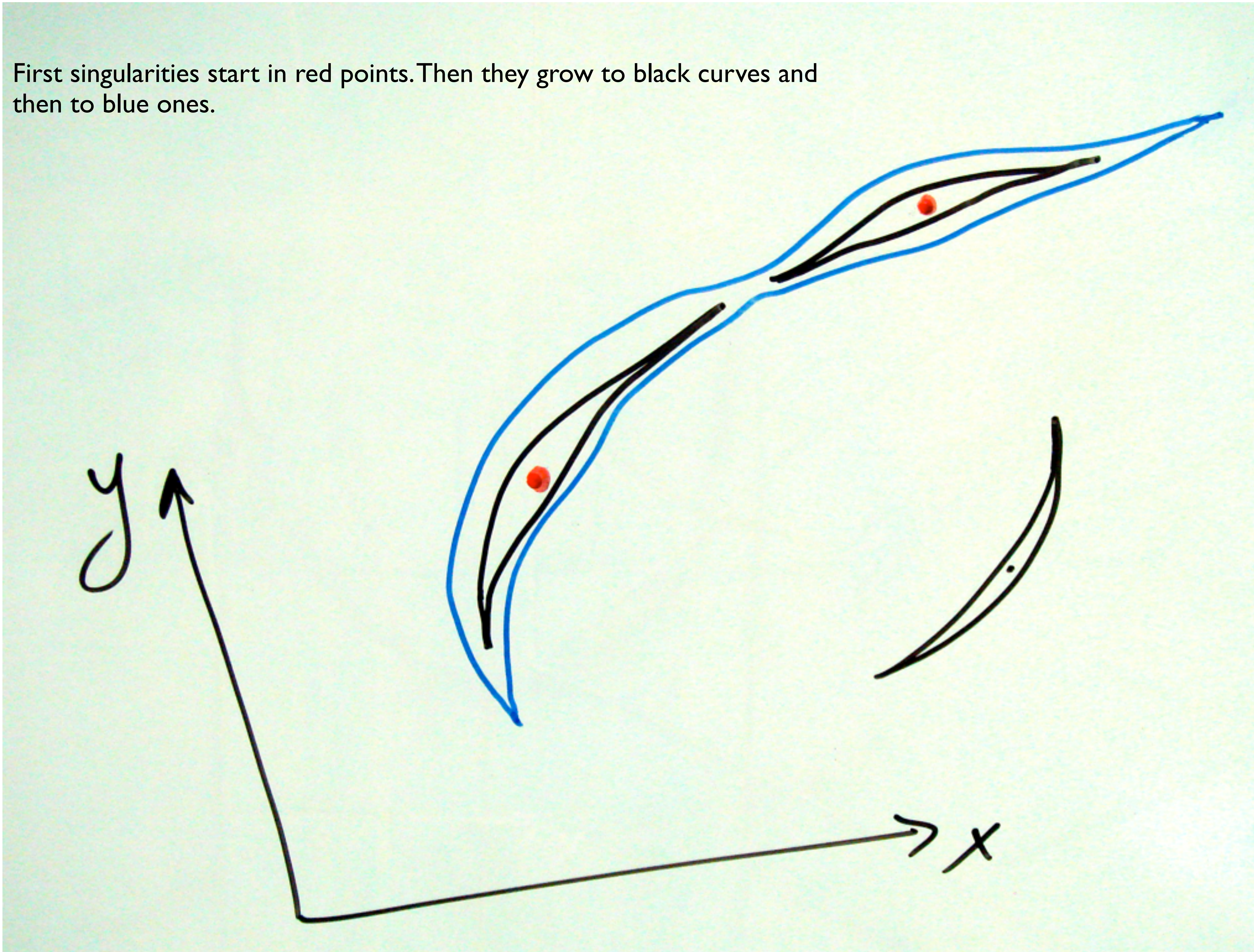
In reality, particles will move away from the center for a short while, then turn around and fall back. This produces a more complicated picture illustrated below. Each folder in the velocity space produces two caustics in real space. At each point in real space there are odd number of flows: 1,3, 5...



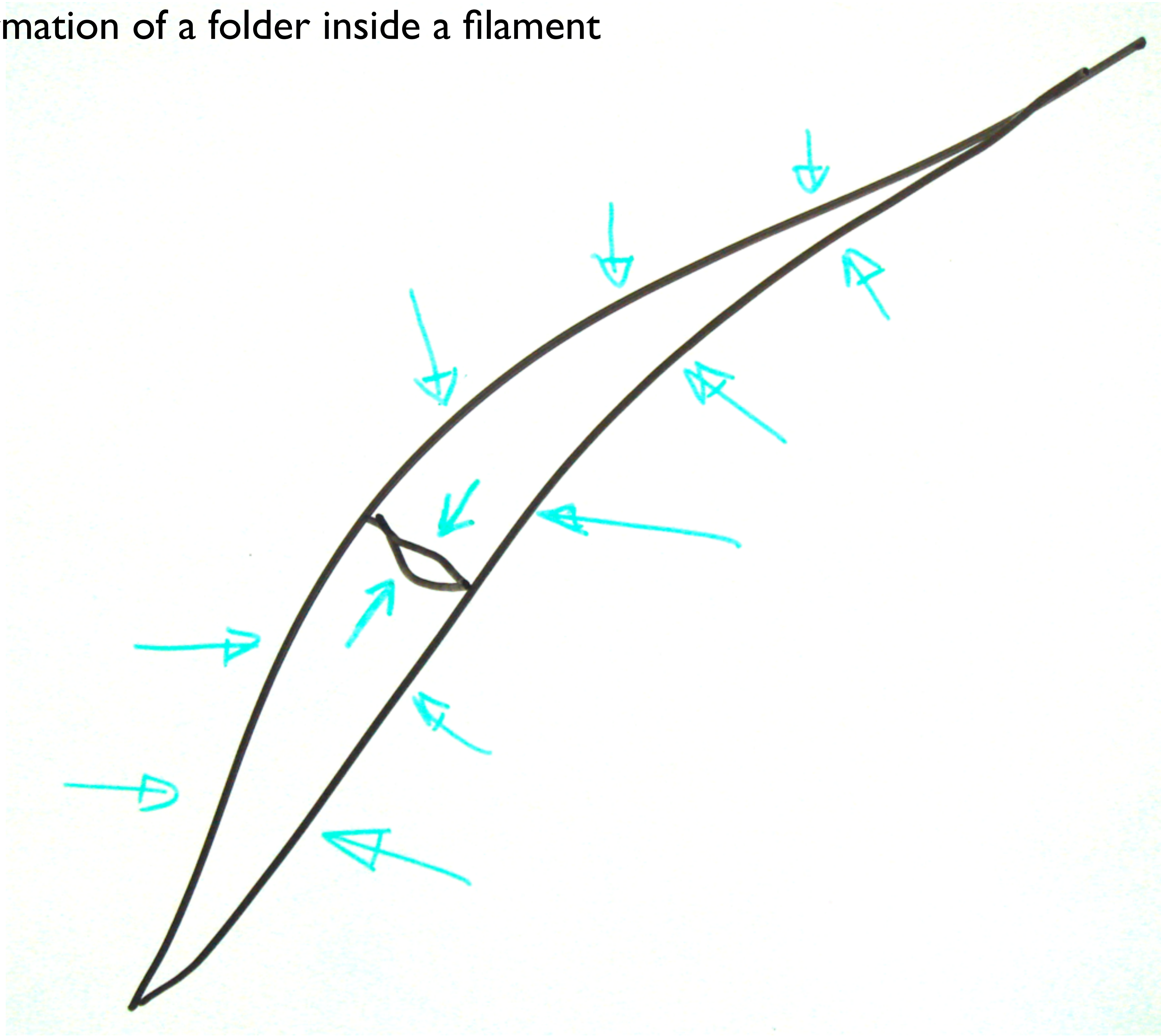
In 2 dimensions:



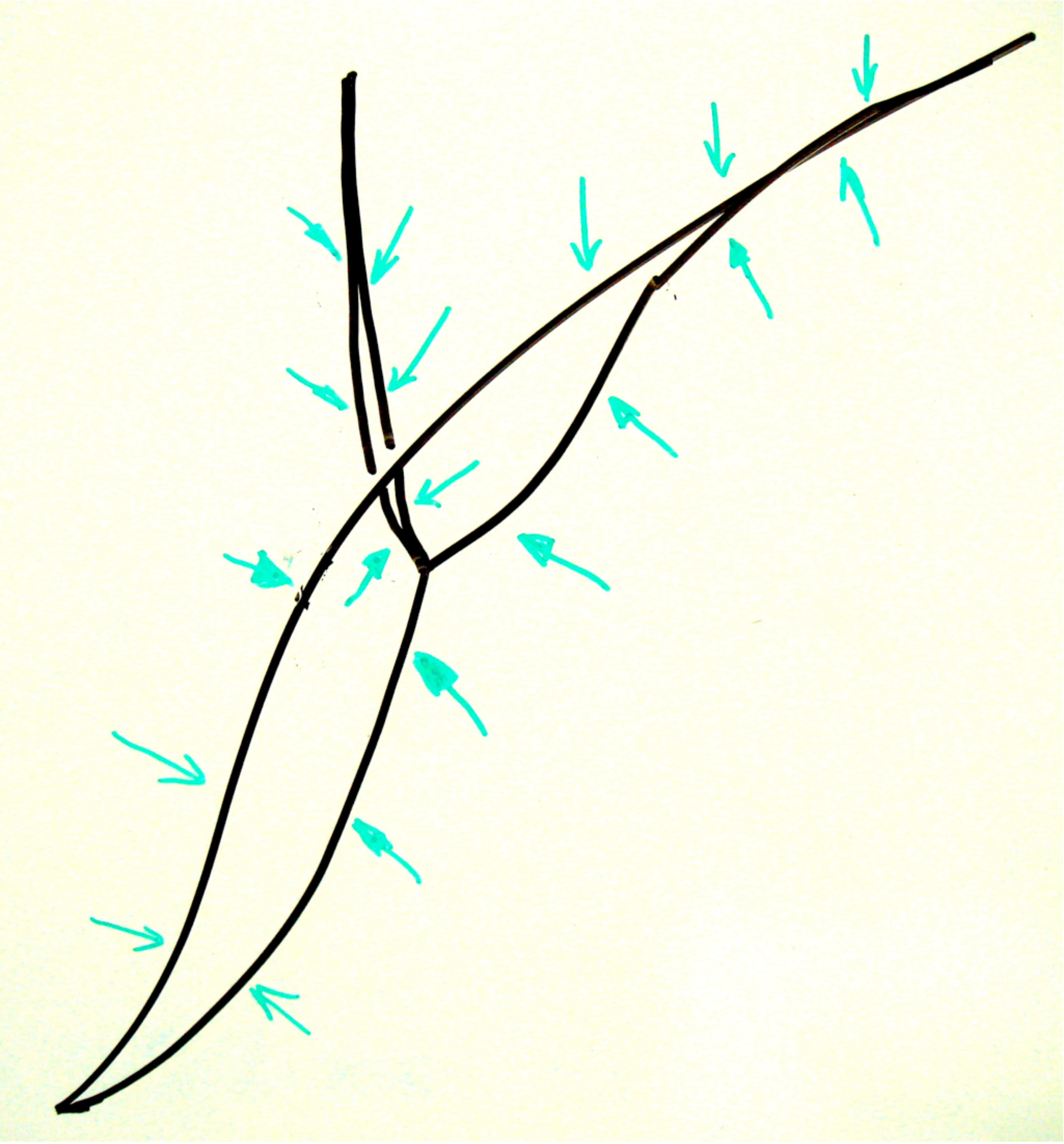
In 2 dimensions: how filaments join - no crossing. They continue smoothly from one to another.



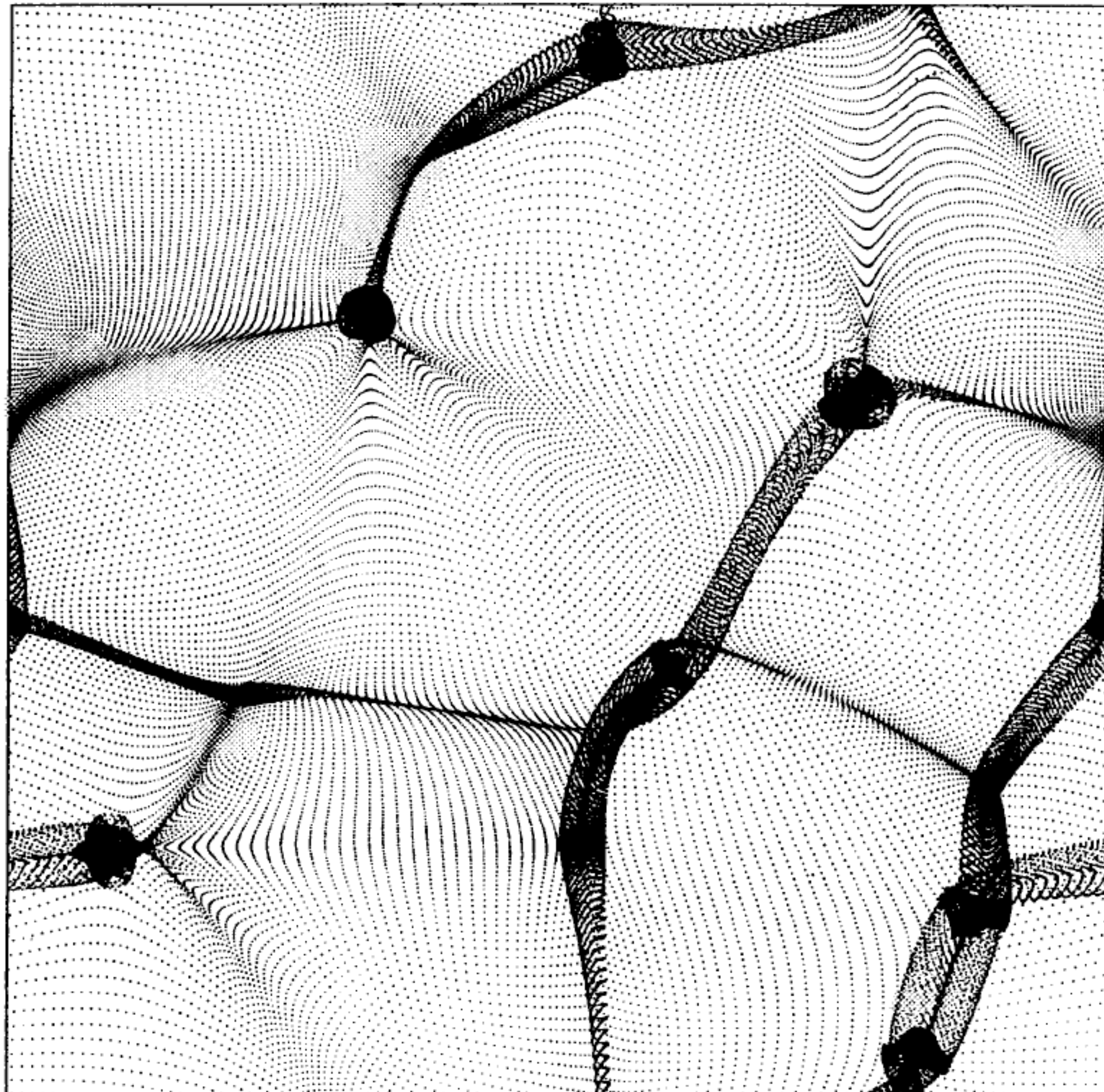
In 2 dimensions: formation of a folder inside a filament



In 2 dimensions: branching filaments due to converging flow along filaments

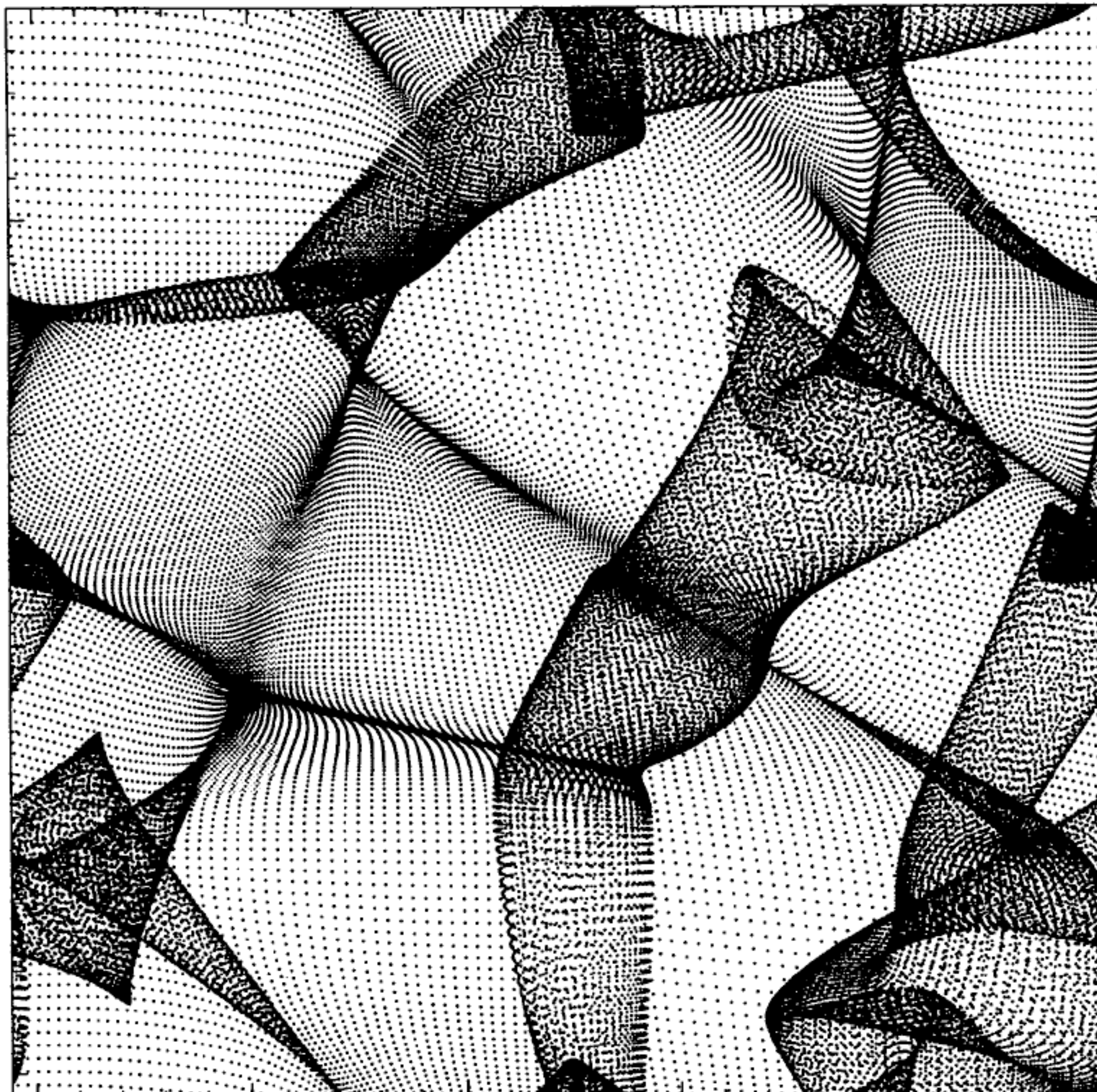


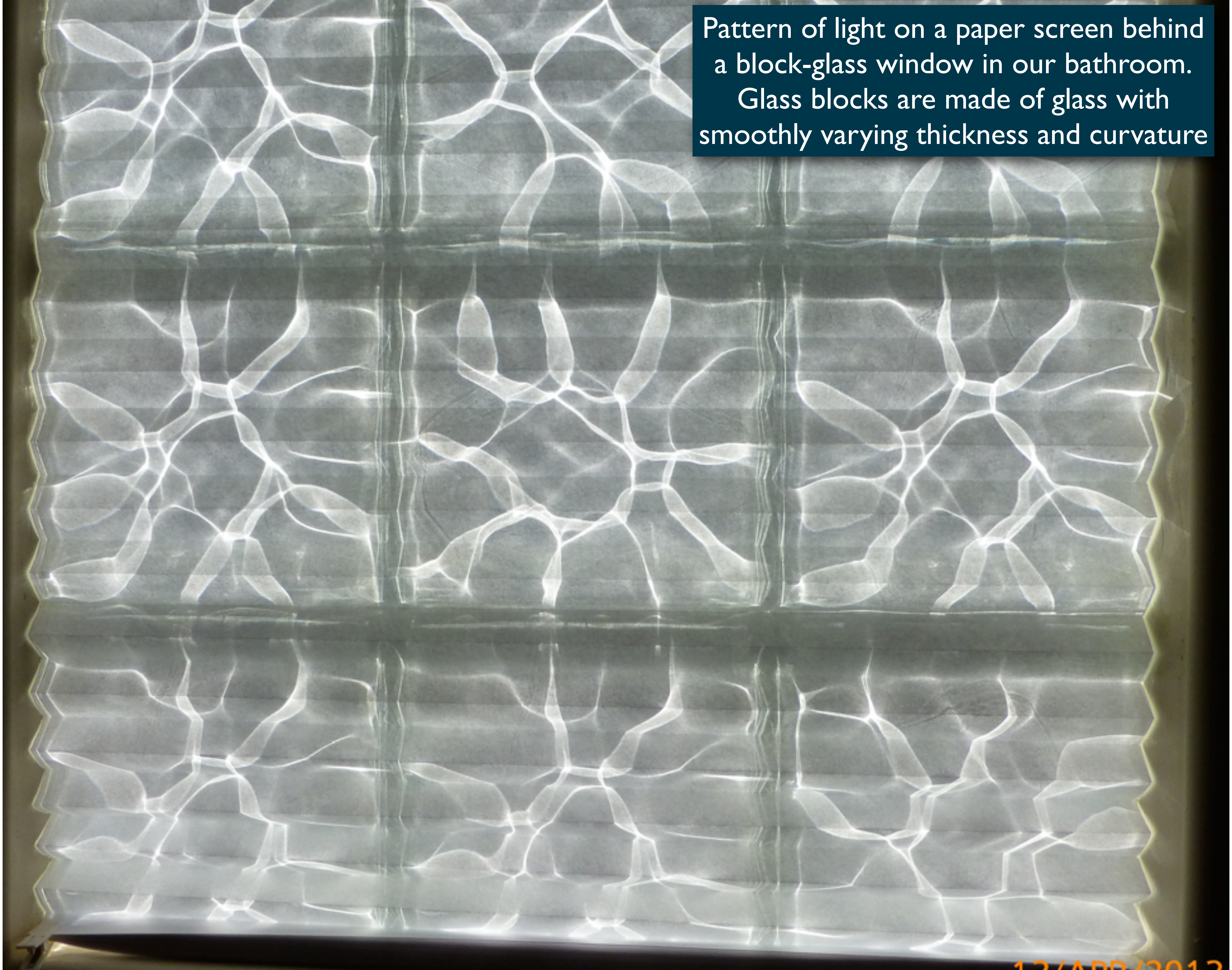
Formation of caustics in a simplified 2D model, which has only very few long-wave harmonics



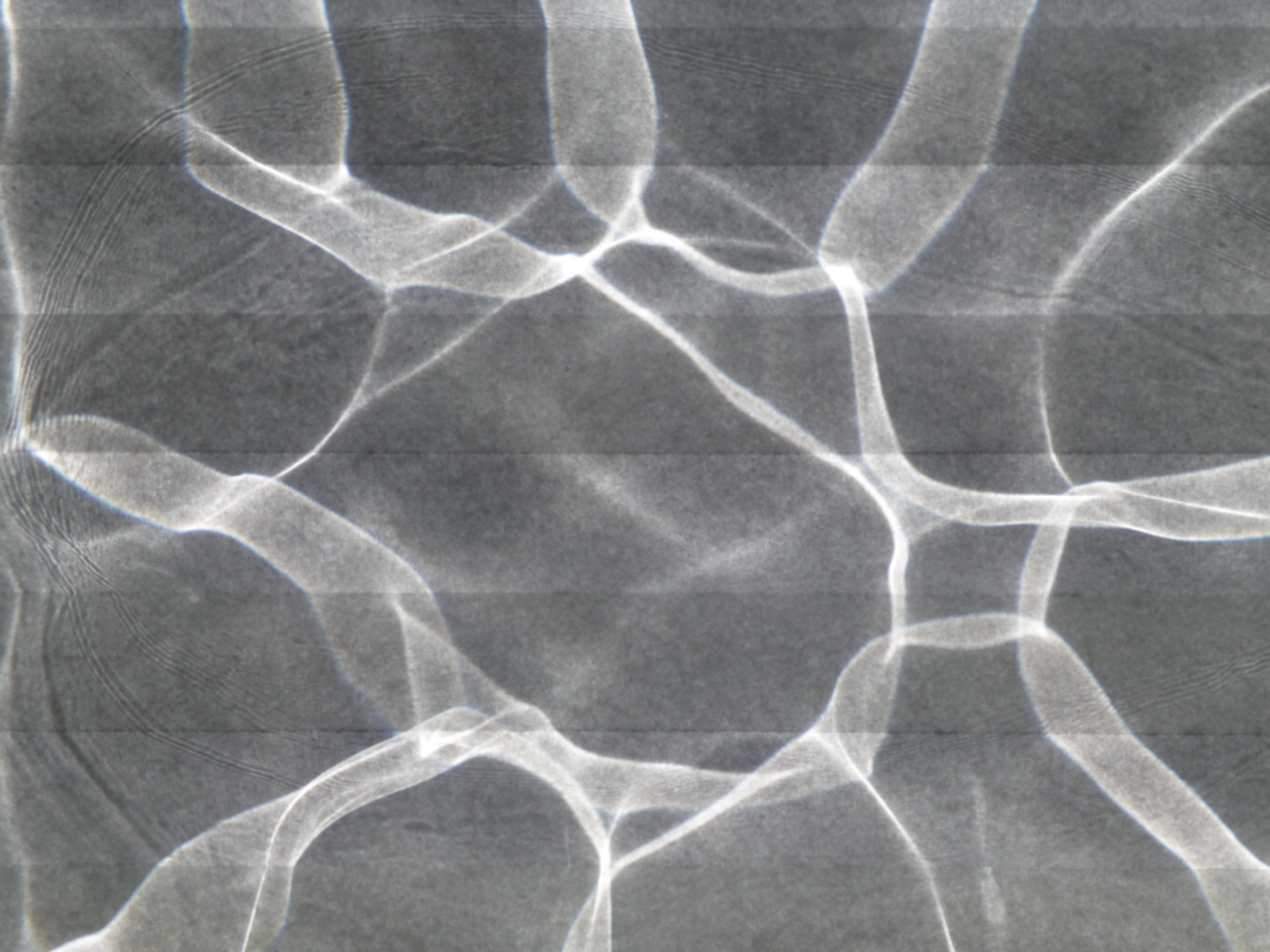
Sahni et al 1994

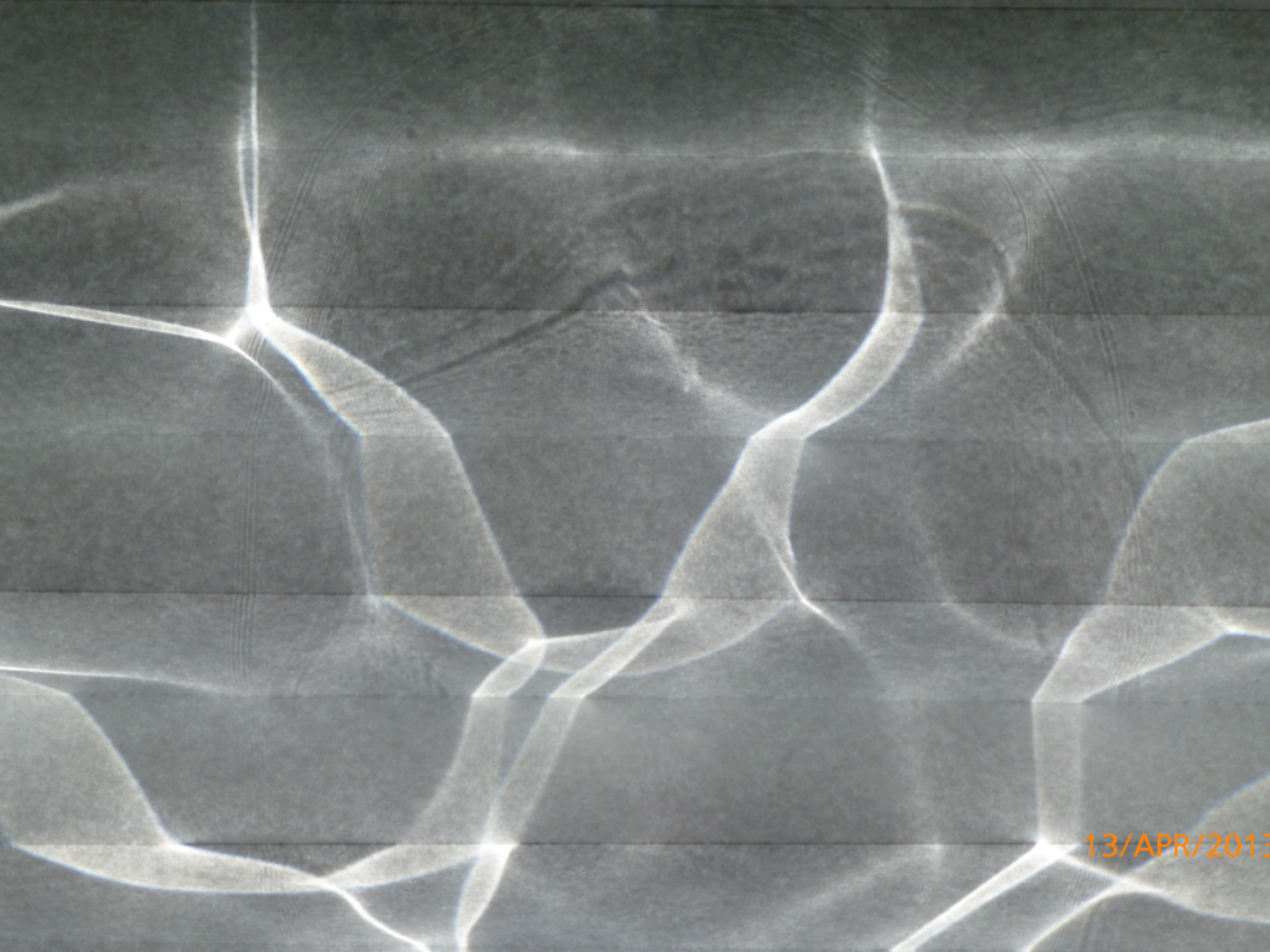
At later moments particles start to fly through regions of high density. This is where the Zeldovich approximation badly fails.





Pattern of light on a paper screen behind a block-glass window in our bathroom. Glass blocks are made of glass with smoothly varying thickness and curvature

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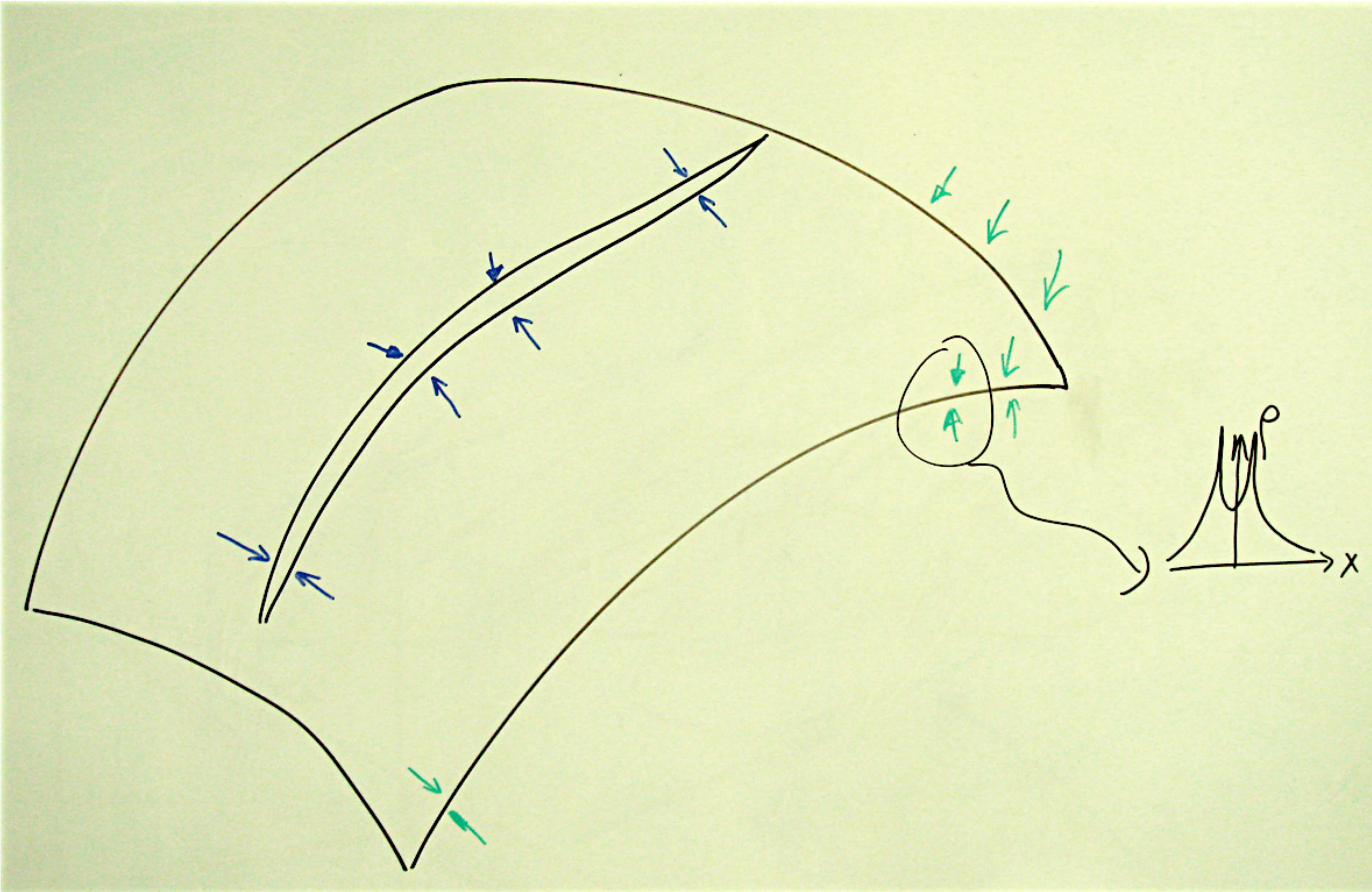
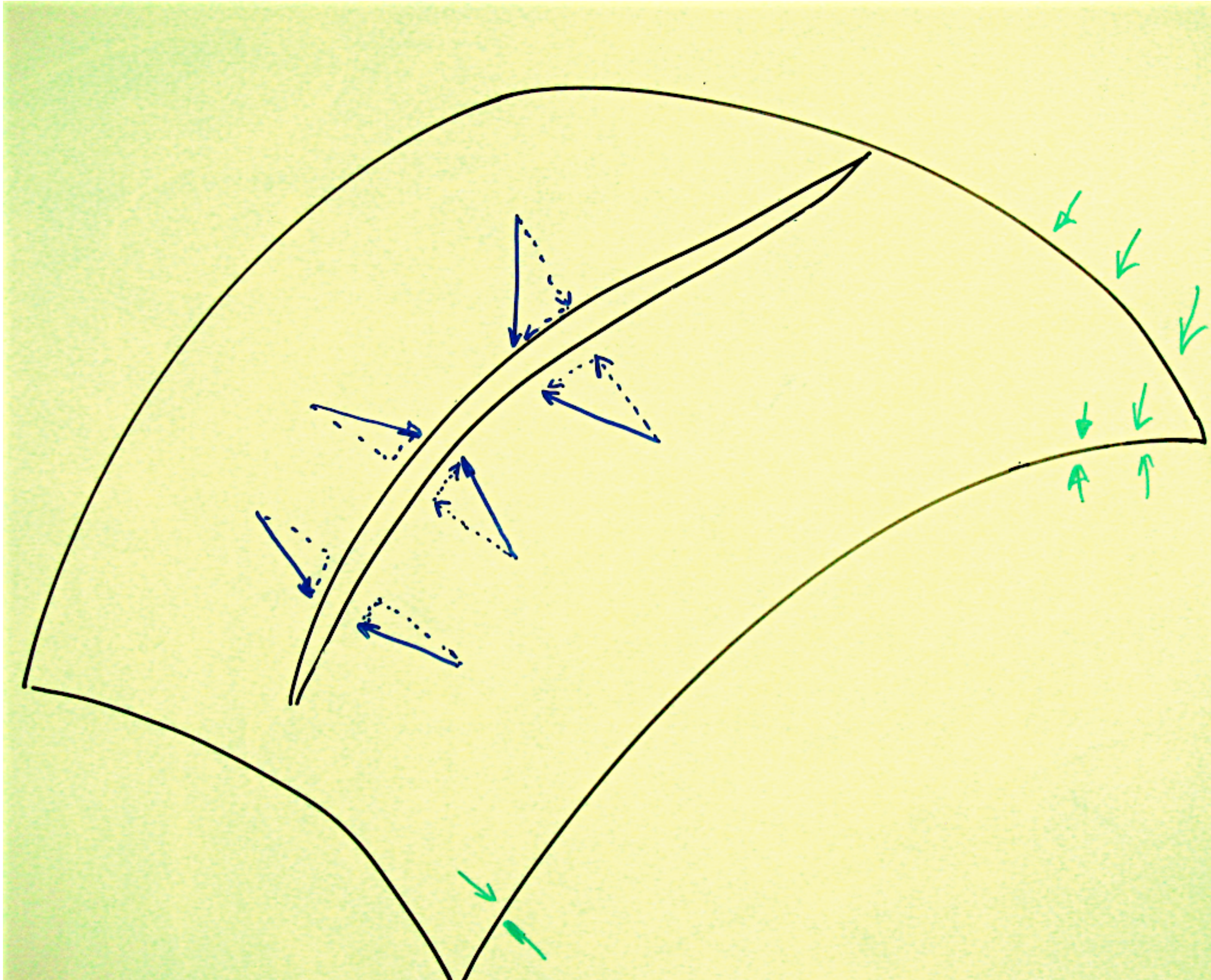
3 dimensions:



3 dimensions:



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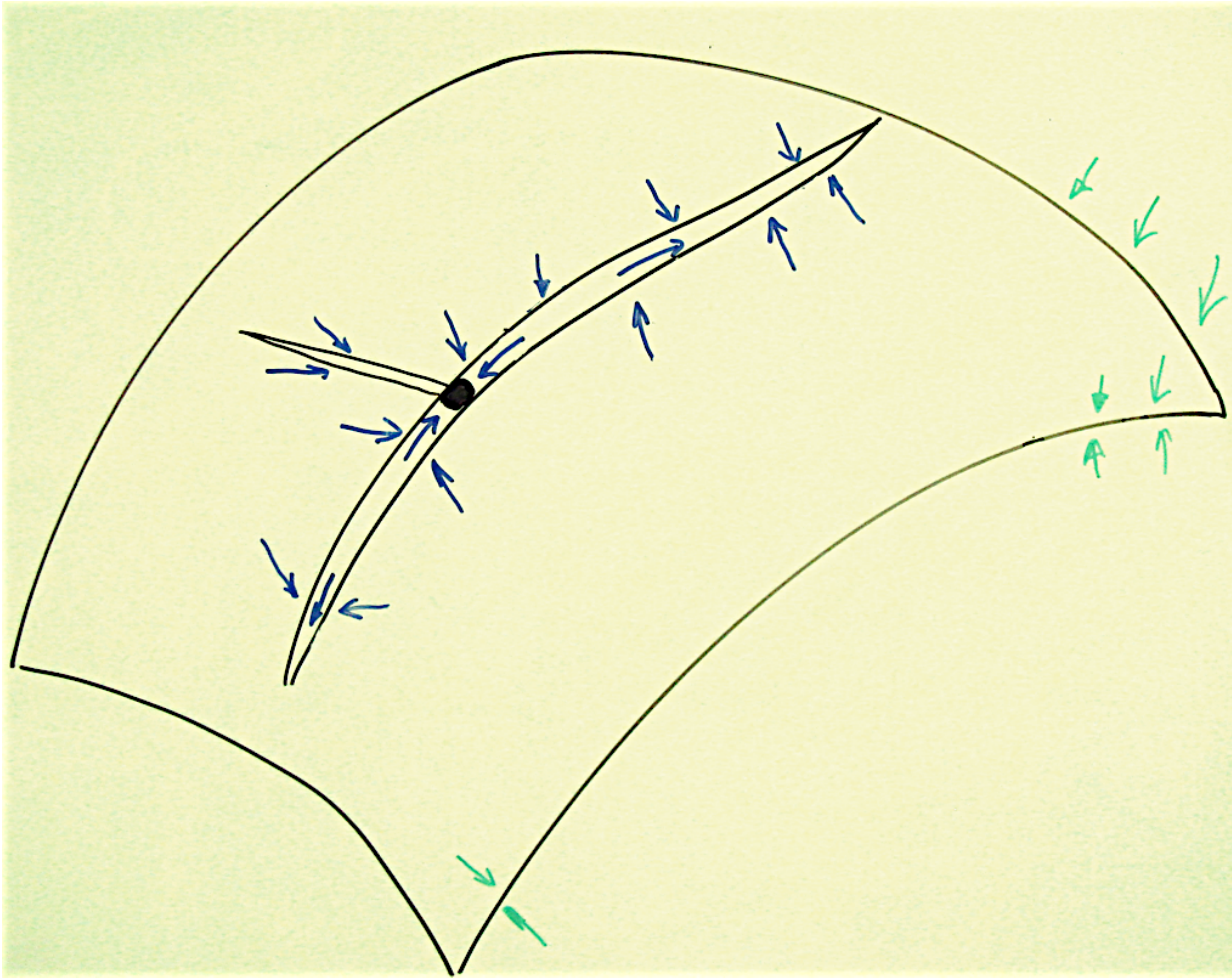
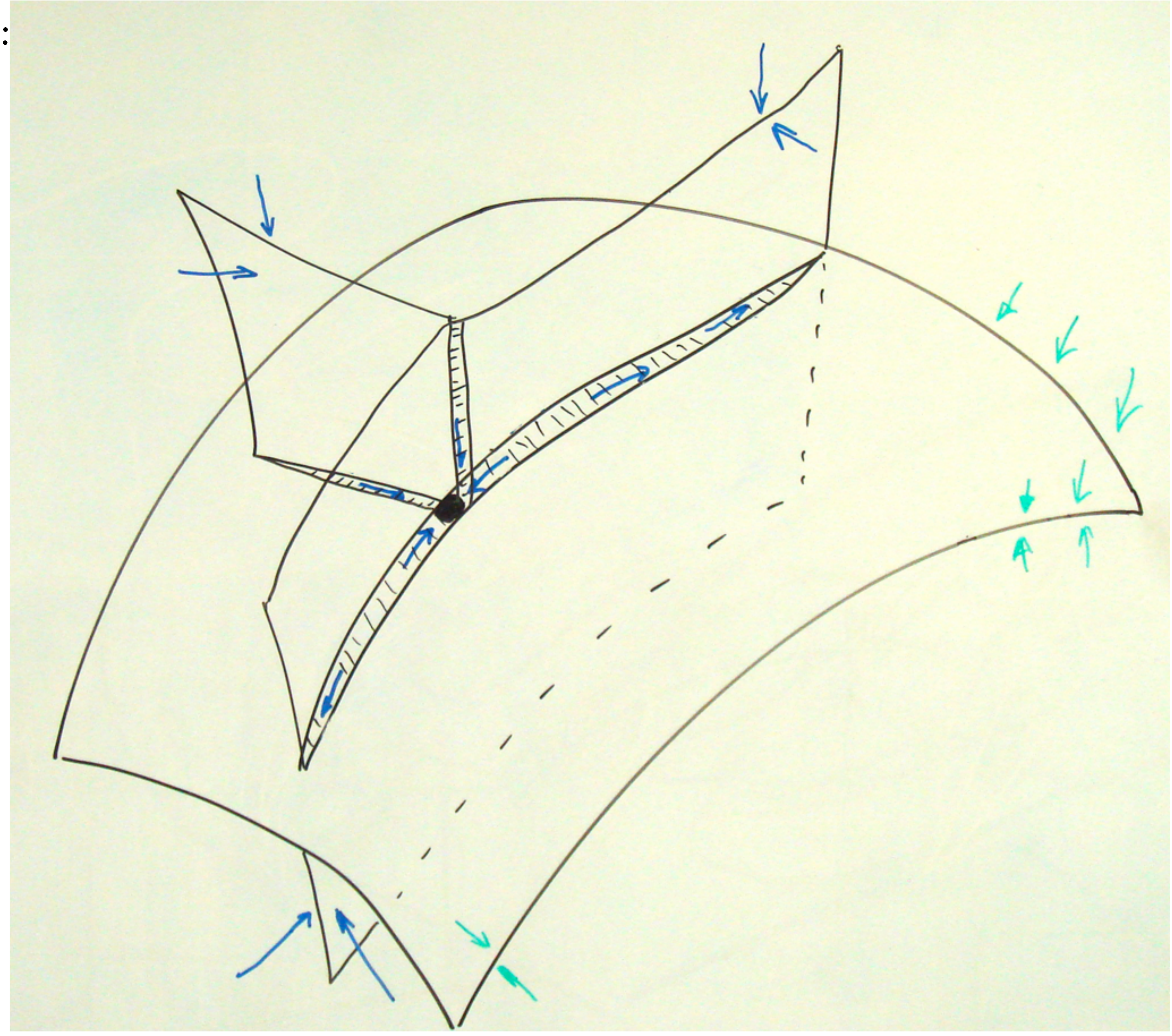


3 dimensions:



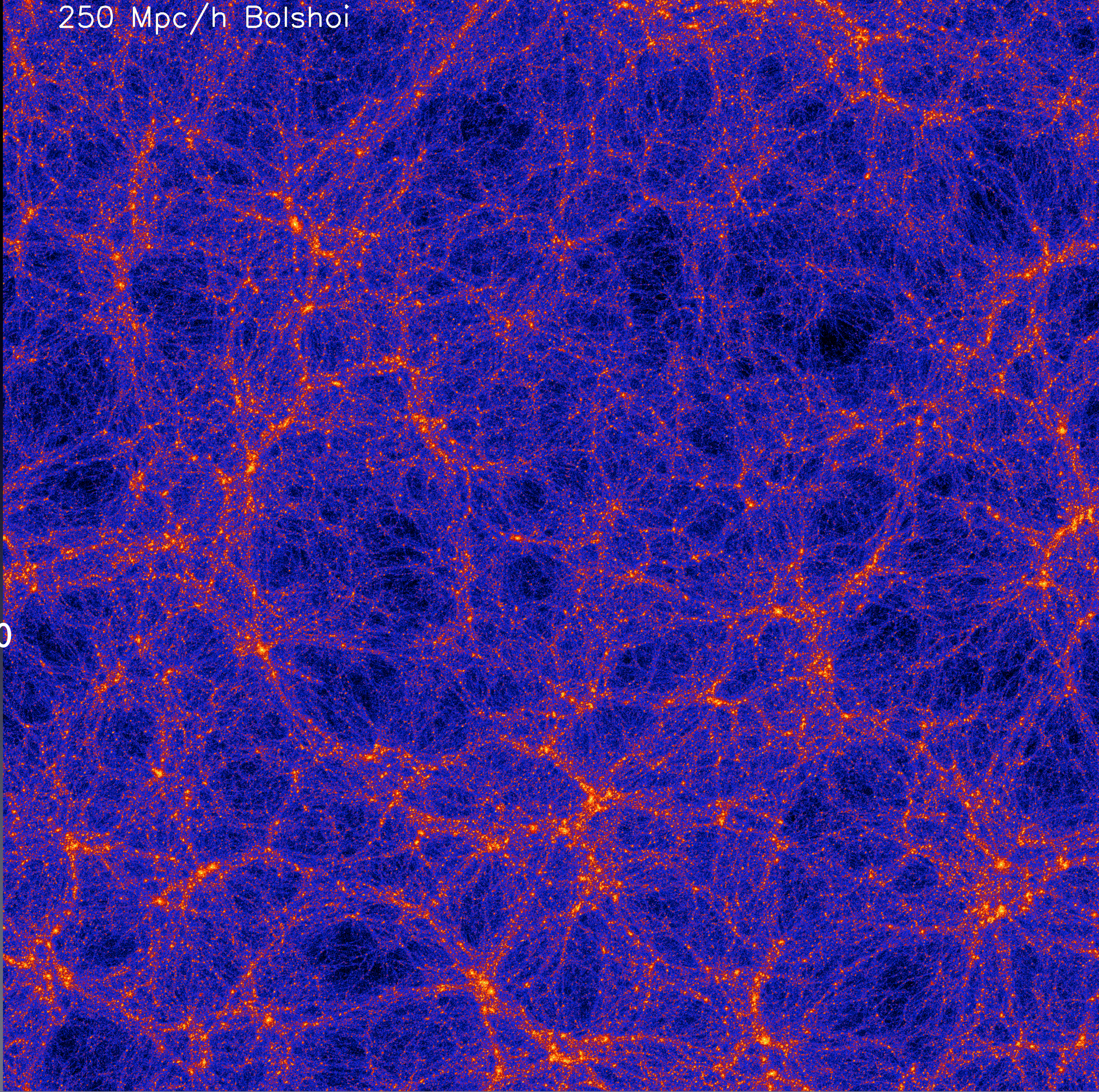
The Bolshoi simulation

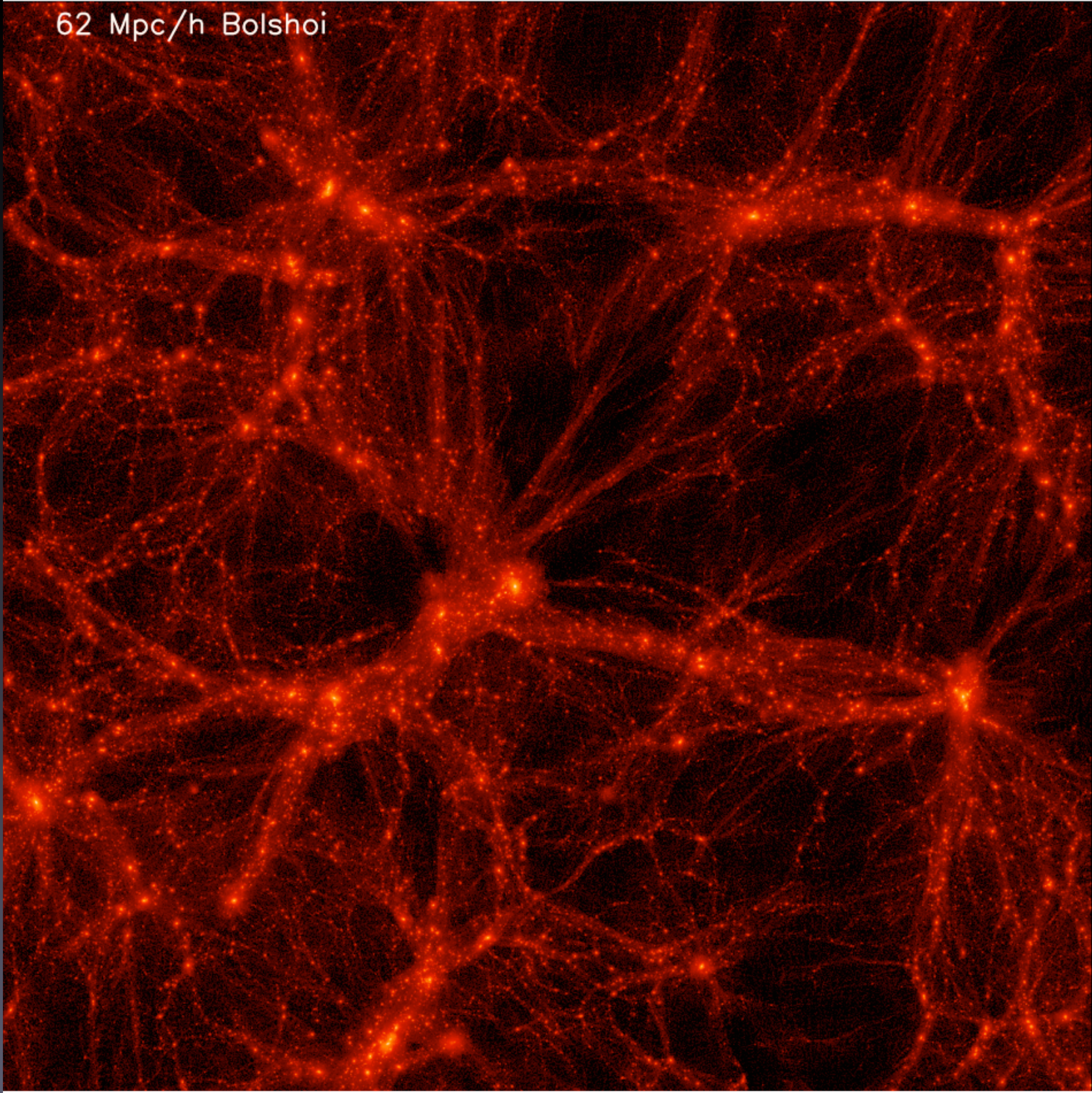
ART code

250Mpc/h Box
LCDM
s8 = 0.82
h = 0.70
8G particles
1kpc/h force resolution
1e8 Msun/h mass res

dynamical range 262,000
time-steps = 400,000

NASA AMES
supercomputing center
Pleiades computer
13824 cores
12TB RAM
75TB disk storage
6M cpu hrs
18 days wall-clock time




62 Mpc/h Bolshoi