Initial spectrum of fluctuations, which was produced during the inflation, is distorted during the evolution of the Universe by different processes.
If $\delta_k^2$ is the square of amplitude of fluctuations at given wavevector $\vec{K}$,

Then we can write: $$\delta_k^2 = A\, k^n\, T(k,t)$$

Where A is a normalization constant, $K^n$ is the initial spectrum of perturbations, and
T(k,t) is the transfer function.

One important case: n=1 is called the Harrison-Zeldovich spectrum.

For some cases we can disentangle the dependencies on k and t. For example, for LCDM or open CDM models the shape of the spectrum did not change much after the recombination : $$T(k,t) = T(k)\, D^2(t)$$
Where D(t) is the growth-factor of fluctuations. Models with hot neutrinos or warm dark matter with some rms velocities of dark matter particles may still experience late changes in growth of perturbations.

**Spectrum of velocities.**
Let's find the relation between perturbations in density and perturbations in peculiar velocity. We are dealing with growing mode, for with there is a unique relation.

$$\begin{cases} \dfrac{\partial \vec{v}}{\partial t} + \dfrac{\dot{a}}{a}\vec{v} = \vec{g} = -\dfrac{\nabla \varphi}{a} \\ \dot{\delta} + \dfrac{1}{a}\nabla \vec{v} = 0 \end{cases}$$

$$\delta(\vec{x}) = \frac{V}{(2\pi)^3}\int d^3k\, \delta_{\vec{k}}\, e^{-i\vec{k}\vec{x}}$$

$$\vec{v}(\vec{x}) = \frac{V}{(2\pi)^3}\int d^3k\, \vec{v}_{\vec{k}}\, e^{-i\vec{k}\vec{x}}$$

Take a plane wave:
$$\vec{v} = \vec{v}_{\vec{k}}\, e^{-i\vec{k}\vec{x}}, \quad \delta = \delta_{\vec{k}}\, e^{-i\vec{k}\vec{x}}$$

Then: $$\nabla \vec{v} = -i(\vec{k}\cdot\vec{v}_k)\exp(-i\vec{k}\cdot\vec{x})$$

Now the continuity equation can be written in the form:
$$\dot{\delta}_{\vec{k}} + \frac{(-i\vec{k}\vec{v}_k)}{a} = 0$$

For growing mode we have derived the relation: $\vec{v} = a\dfrac{\partial}{\partial t}\left(\dfrac{\vec{g}}{4\pi G\rho a}\right) \Rightarrow$
$$\Rightarrow \vec{v} \parallel \vec{g} \Rightarrow \vec{v} \parallel \vec{k}$$

Thus, $(\vec{K}\vec{v}_K) = K v_K$

$$v_K = a\,\frac{\dot{\delta}_K}{iK}$$

Write density perturbation in the form:

$$\dot{\delta} = \frac{a}{\delta}\frac{d\delta}{da}\cdot\frac{\dot{a}}{a}\delta = H f(\Omega)\delta$$

$$f(\Omega) \equiv \frac{a}{\delta}\frac{d\delta}{da}$$

We get: $v_K = \frac{Ha}{iK} f(\Omega)\delta_K \Leftarrow$ 90degrees rotation relative to $\delta_K$

Power spectrum of velocities is:

$$v_K^2 = H^2 a^2 f^2(\Omega)\,\frac{\delta_K^2}{K^2}$$

Consider a filed of density perturbations in a volume $V$

$$\delta(\vec{x}) = \frac{\rho(\vec{x}) - \rho_b}{\rho_b}$$

The average of the density contrast is equal to zero:

$$\langle \delta(\vec{x}) \rangle = 0$$

Let's find the dispersion of the density contrast: $\langle \delta^2(\vec{x}) \rangle$
Decompose the density contrast into the Fourier spectrum

$$\delta(\vec{x}) = \frac{V}{(2\pi)^3} \int d^3k \, \delta_{\vec{k}} \, e^{-i\vec{k}\vec{x}}$$

$$\delta_{\vec{k}} = \frac{1}{V} \int d^3x \, \delta(\vec{x}) \, e^{i\vec{k}\cdot\vec{x}}$$

Find the dispersion:

$$\langle \delta^2(\vec{x}) \rangle = \frac{1}{V} \int d^3x \, \delta^2(\vec{x}) = \frac{1}{V} \int d^3x \, \delta(x) \delta(x) =$$

$$= \int \frac{d^3x}{V} \, V \int \frac{d^3k}{(2\pi)^3} \delta_{\vec{k}} e^{-i\vec{k}\vec{x}} \, V \int \frac{d^3k'}{(2\pi)^2} \delta^*_{\vec{k}'} e^{i\vec{k}'\vec{x}} = \qquad \Bigg\| \; \delta_{-\vec{k}} = \delta^*_{\vec{k}}$$

$$= \int \frac{d^3x}{V} \, V^2 \int \frac{d^3k \, d^3k'}{(2\pi)^6} \delta_{\vec{k}} \delta^*_{\vec{k}'} \, e^{-i(\vec{k}-\vec{k}')\vec{x}}$$

Change the order of integration:

$$\langle \delta^2 \rangle = \int \frac{d^3k \, d^3k'}{(2\pi)^6} \delta_{\vec{k}} \delta^*_{\vec{k}'} \underbrace{V^2 \int \frac{d^3x}{V} e^{-i(\vec{k}-\vec{k}')\vec{x}}}_{\delta(\vec{k}-\vec{k}')}$$

$$= V \int \frac{d^3k}{(2\pi)^3} |\delta_{\vec{k}}|^2 = \frac{V}{(2\pi)^3} \int_0^\infty 4\pi k^2 dk \, |\delta_k|^2$$

Thus, we get:

$$\langle \delta^2 \rangle = \frac{V}{2\pi^2} \int_0^\infty k^2 \delta_k^2 \, dk$$

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We define the **power spectrum** as

Here the averaging is done over all
Waves with given k and over the whole space

$$P(k) = \langle |\delta_{\vec{k}}|^2 \rangle$$

**Correlation function** is defined as

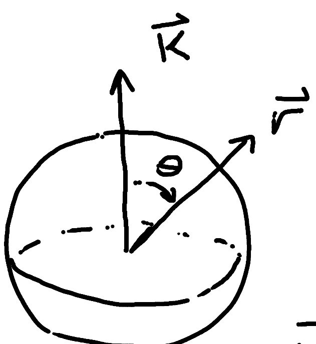
$$\xi(r) = \langle \delta(\vec{x})\delta(\vec{x}+\vec{r}) \rangle$$

The averaging is done for the whole
volume and over angles of vector $\vec{r}$

$$\xi(r) = \frac{1}{4\pi}\int d\Omega \int \frac{d^3x}{V} \int \frac{Vd^3k}{(2\pi)^3}\delta_{\vec{k}} e^{-i\vec{k}\vec{x}} \int \frac{Vd^3k'}{(2\pi)^3}\delta^*_{\vec{k}'} e^{i\vec{k}'(\vec{x}+\vec{r})} =$$

$$= \frac{1}{4\pi}\int d(\cos\theta)d\varphi \int \frac{Vd^3k}{(2\pi)^3}\delta_{\vec{k}} e^{i\vec{k}\vec{r}} \delta^*_{\vec{k}} =$$

$$= \int \frac{Vd^3k}{(2\pi)^3}\delta_{\vec{k}}\delta^*_{\vec{k}} \frac{1}{4\pi}\int d(\cos\theta)d\varphi \, e^{i\vec{k}\vec{r}}$$




imaginary

$$e^{i\vec{k}\vec{r}} = \cos(kr\cos\theta) + \cancel{i\sin(kr\cos\theta)}$$

$$\frac{1}{4\pi}\int_{-1}^{1} dx \int_0^{2\pi} d\varphi \cos(krx) = \frac{\sin(kr)}{kr}$$

Thus, we get the relation between the correlation function and the power spectrum:

$$\xi(r) = \frac{1}{2\pi^2}\int_0^\infty k^2 dk \, \frac{\sin kr}{kr} P(k)$$

There is an inverse relation:

$$P(k) = 4\pi\int r^2 dr \, \xi(r) \frac{\sin kr}{kr}$$

# Filters



We need to find a way to deal with the random fields and get physical statistics such as mass variations. For that we introduce filters, which define some scale of smoothing of the density fields.

Example: **top-hat filter**

Window function is defined as $$W(r) = \begin{cases} 1, & r \le r_0 \\ 0, & r > r_0 \end{cases}$$

Volume and mass of the filter are: