

The Universe is not uniform. We have ignored this when we talked about FRW metric and when we discuss physics of early Universe such as Big Bang Nucleosynthesis or neutrino freeze-out (when neutrino decouple from the rest of the matter). There is a reason why we treated the Universe as homogeneous: the deviations from homogeneity are small and they were even smaller in the past.

There numerous issues related with perturbations. Big Bang itself cannot explain how the fluctuations formed. There is a natural source of fluctuations: statistical fluctuations in a medium, which consists of discrete particles. The amplitude of those fluctuations is roughly  $1/\sqrt{N}$ , where  $N$  is the number of particles in some given volume. There are two problems with those fluctuations. First, their amplitude is very small. For example, consider a cluster of galaxies with mass about  $1e15$  Solar mass. Calculate the number of protons and take square root of it. This is what we expect from statistical mechanics. Second, the fluctuations (small or very small) grow relatively slow. This is due to the expansion of the Universe. In the absence of expansion the fluctuations grow exponentially:

$$\delta \propto e^{t/t_{dyn}}, \quad t_{dyn} \approx \frac{1}{\sqrt{4\pi G\rho}}$$

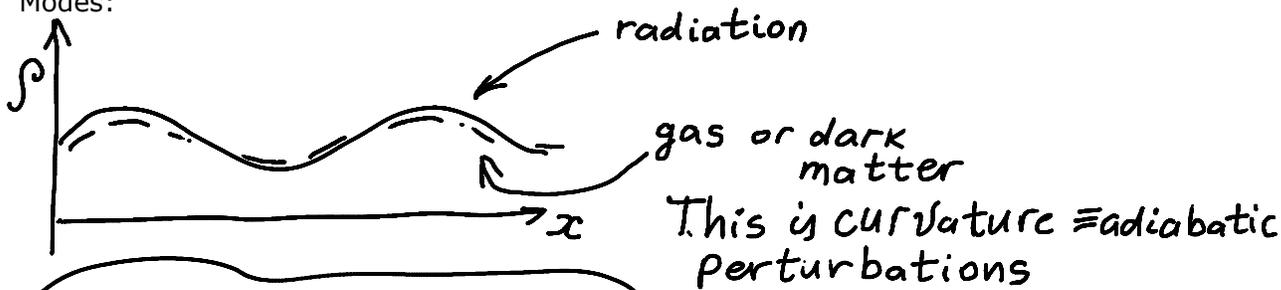
Here  $\rho$  is the density and  $t_{dyn}$  is the dynamical time scale. Unfortunately, the

fluctuations grow much slower: only as a power-law  $\delta \propto a^\sigma$ ,  $\sigma \approx 1-2$  and statistical fluctuations do not

play any role as an origin of fluctuations. Thus, we need something else. *So far, the only explanation for the origin of fluctuations is coming from Inflation.*

Regardless their origin, fluctuations can be decomposed into MODES (components). In addition, we need to pay attention to different physical components: perturbations in dark matter, in gas, or radiation are different and they evolve differently.

Modes:

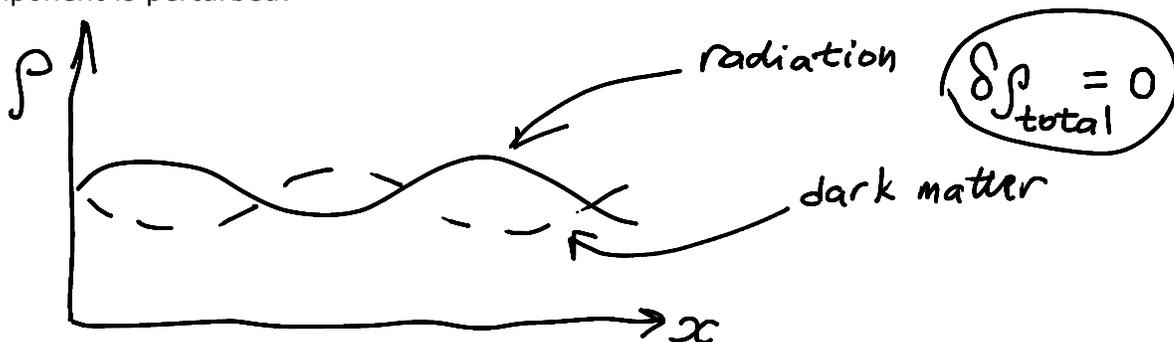


$$\rho(\vec{x}) = \rho_{background} + \delta\rho(\vec{x})$$

$$\delta \equiv \frac{\delta\rho}{\rho_{backgr}}$$

In this mode the metric is perturbed and The amplitude of perturbations initially is the same for all different mass component.

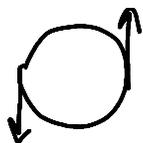
Isocurvature or isothermal fluctuations: total density initially is not perturbed, but each component is perturbed:



tensor  $\equiv$  gravity waves

$$\Rightarrow \delta v = 0, \delta \rho_{total} = 0, g_{\alpha\beta} = \bar{g}_{\alpha\beta} + h_{\alpha\beta}$$

$$h_{\alpha\beta} \propto 1/a$$



Vector perturbations:

$$\text{div } \vec{v} = 0, \delta \rho_{total} = 0$$

$$v \propto 1/a$$

Different notions:

Decaying and growing modes. Equations of evolution of perturbations are second order ODE. Those have two solutions: increasing amplitude and declining amplitude. We ignore the decaying modes almost everywhere except during transitions between different regimes. Here we need to match solutions from one regime to another. This involves both modes.

Perturbations inside and outside the horizon evolve typically differently. We will consider two limiting regimes: much longer and much shorter than the distance to the horizon.

Perturbations in matter grow differently before and after the epoch of equality (moment when density of relativistic and nonrelativistic particles equals).

Around the moment of recombination there interesting physical processes (e.g., Silk dumping), which affect perturbations in baryons.

When the Universe starts to accelerate at late stages (due to the cosmological constant or dark energy), fluctuations start to grow very slowly.

The rigorous approach to the evolution of perturbations is to trace the evolution of perturbations in metric caused (and coupled) by perturbations in the energy-stress tensor. Here is a very short story.

We impose small perturbations in metric:  $g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu}$

Where  $h_{\mu\nu}$  are small perturbations.

Since the metric is a symmetric 4x4 matrix, we have 10 independent functions. There is a freedom of using the gauge: some variations in metric are just coordinate transformations, not real physical perturbations. There are 4 free functions for those non-perturbative adjustments. Of the remaining 6 independent functions:

- 2 describe scalar perturbations (trace of the metric and the spatial curvature)
- 1 describes vector (rotation) perturbation (2 components)
- 1 tensor (gravity wave) (2 components)

We are not interested in the vector modes: they die out as the Universe expands. The gravity waves lose their energy in the same way as the radiation. Once the grav.waves enter the horizon, they start to die out.

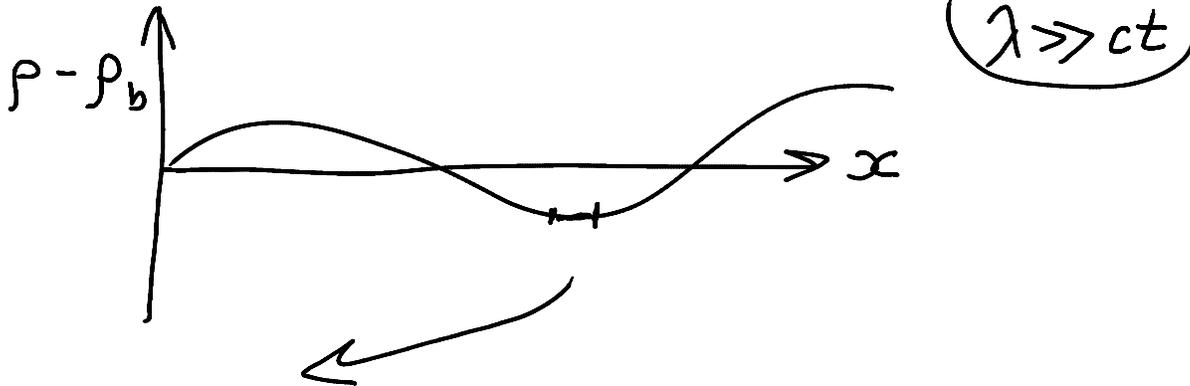
We are left with two scalar components. One of them is growing and another is decaying.

This is still a lot because we have different physical components (radiation, dark matter, gas) and we have different regimes of evolution.

In the linear stage of evolution different modes evolve independently. Because of the different physical components, we need to solve a system of coupled linear differential equations. There are ways of doing this. Solving the Boltzman equation is the best way. There are analytical approximations, which produce remarkably accurate results when compared with the direct Boltzman solutions.

We will be interested in understanding different physical processes and in putting together the whole picture. When time comes, we will use results of accurate modeling. Equations, which we will derive are accurate for physical processes and regimes, which we will use. For example, the growth of perturbations well inside the horizon give a very accurate description how fluctuations grow at late stages of evolution before they become non-linear.

Adiabatic modes with wavelength much longer than the distance to the horizon. There are two modes: growing and decaying. Here is the derivation for the growing mode:



Each fragment of the wave has size of the horizon at that moment of time. Thus, different parts of the long wave cannot "communicate" with other parts. Each fragment evolves independently as a Friedmann universe with slightly different density, but with the same Hubble constant:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho_b, \quad \rho_b = \text{unperturbed density}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho - \frac{\kappa}{a^2}, \quad \rho = \text{perturbed density}$$

$$\rho = \rho_b + \delta\rho$$

Because the Hubble constant is the same (this selects the fastest growing mode), We get:

$$\frac{8\pi G}{3} \rho - \frac{\kappa}{a^2} = \frac{8\pi G}{3} \rho_b \Rightarrow$$

$$\delta = \frac{\delta\rho}{\rho} = \frac{\kappa}{a^2} \frac{3}{8\pi G} \frac{1}{\rho_b} \propto \frac{1}{a^2 \rho_b}$$

For matter-dominated Universe we get:

$$\rho_b \propto a^{-3} \Rightarrow \delta \propto a \propto t^{2/3}$$

For radiation-dominated Universe:

$$\rho_b \propto a^{-4} \Rightarrow \delta \propto a^2 \propto t$$