Growth of fluctuations II

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- Transfer Function
- Velocities
- Power Spectrum

General equation for growth of perturbations is:

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} = 4\pi G\rho_b\delta + \frac{\nabla^2 P}{\rho_b a^2}$$

Particular cases. The simplest case is ***the flat Universe, wavelength shorter than the horizon, waves longer than the Jeans mass.*** This is also the case of the cold dark matter (negligible random velocities):

$$P=0,\ \Omega_m=1,\ \rho_b=\frac{1}{6\pi G t^2}\propto a^{-3}$$

Friedmann equation is: $$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho = \left(\frac{2}{3t}\right)^2$$

ow the equation for the density contrast can be written as:

$$\ddot{\delta} + \frac{4}{3}\frac{\dot{\delta}}{t} = \frac{2}{3}\frac{\delta}{t^2}$$

olution of this equation is found in the form $\delta = At^n$

his gives: $$n(n-1)+\frac{4}{3}n = \frac{2}{3} \rightarrow n=\frac{2}{3} \text{ or } n=-1$$

hus, the general solution is:

$$\delta(x,t) = \underbrace{A(x)}_{\text{growing}}t^{2/3} + \underbrace{B(x)}_{\text{decaying}}t^{-1}$$

Case: ***flat Universe with cosmological constant***

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda c^2}{3}$$

Solution for the growth of small perturbations is

$$\delta = \frac{1}{X_0}\frac{\sqrt{1+x^3}}{x^{3/2}}\int_0^x \frac{x^{3/2}\,dx}{[1+x^3]^{3/2}}$$

$$\Omega_\Lambda = \Omega_{\Lambda,0}\frac{H_0^2}{H^2}$$

$$x = x_0 a = \left(\frac{\Omega_{\Lambda,0}}{\Omega_0}\right)^{1/3} a$$

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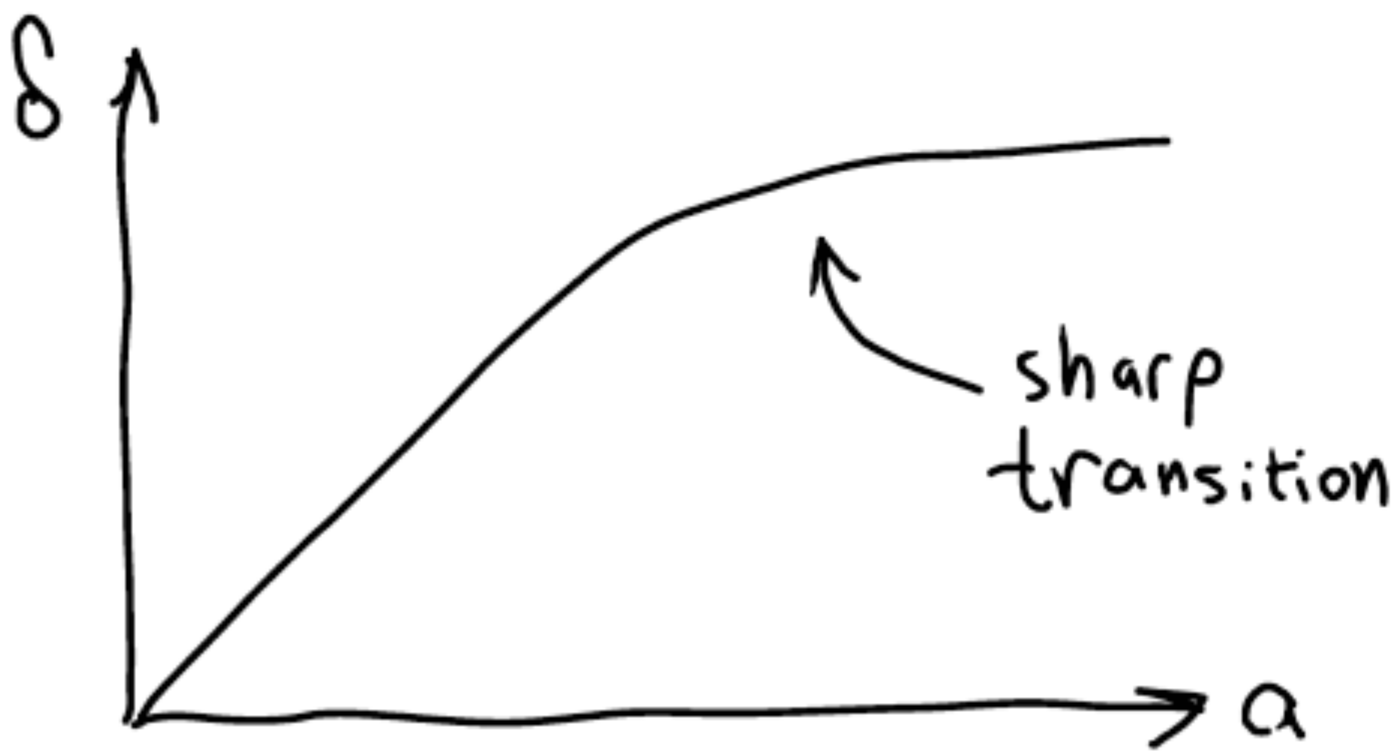
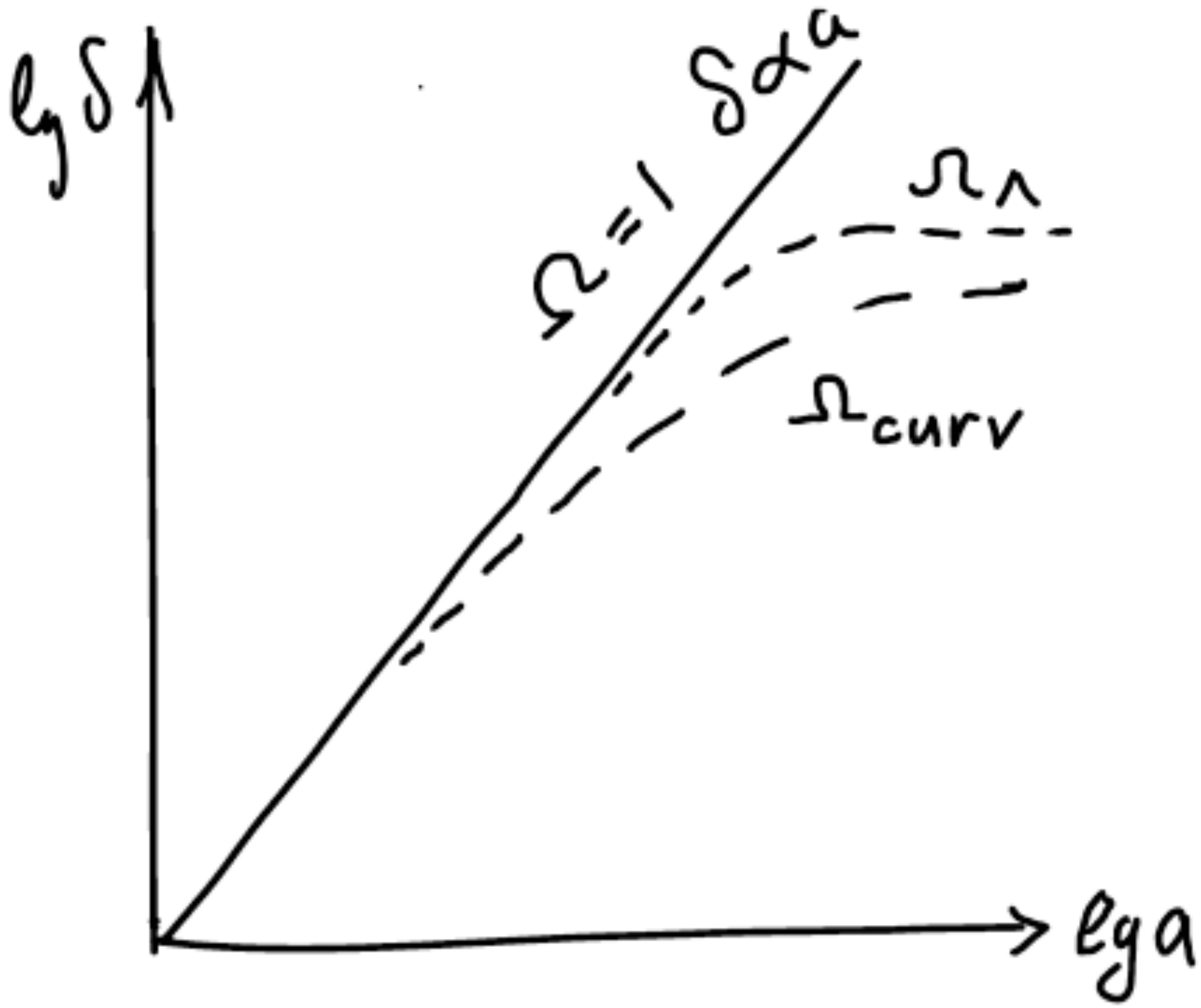
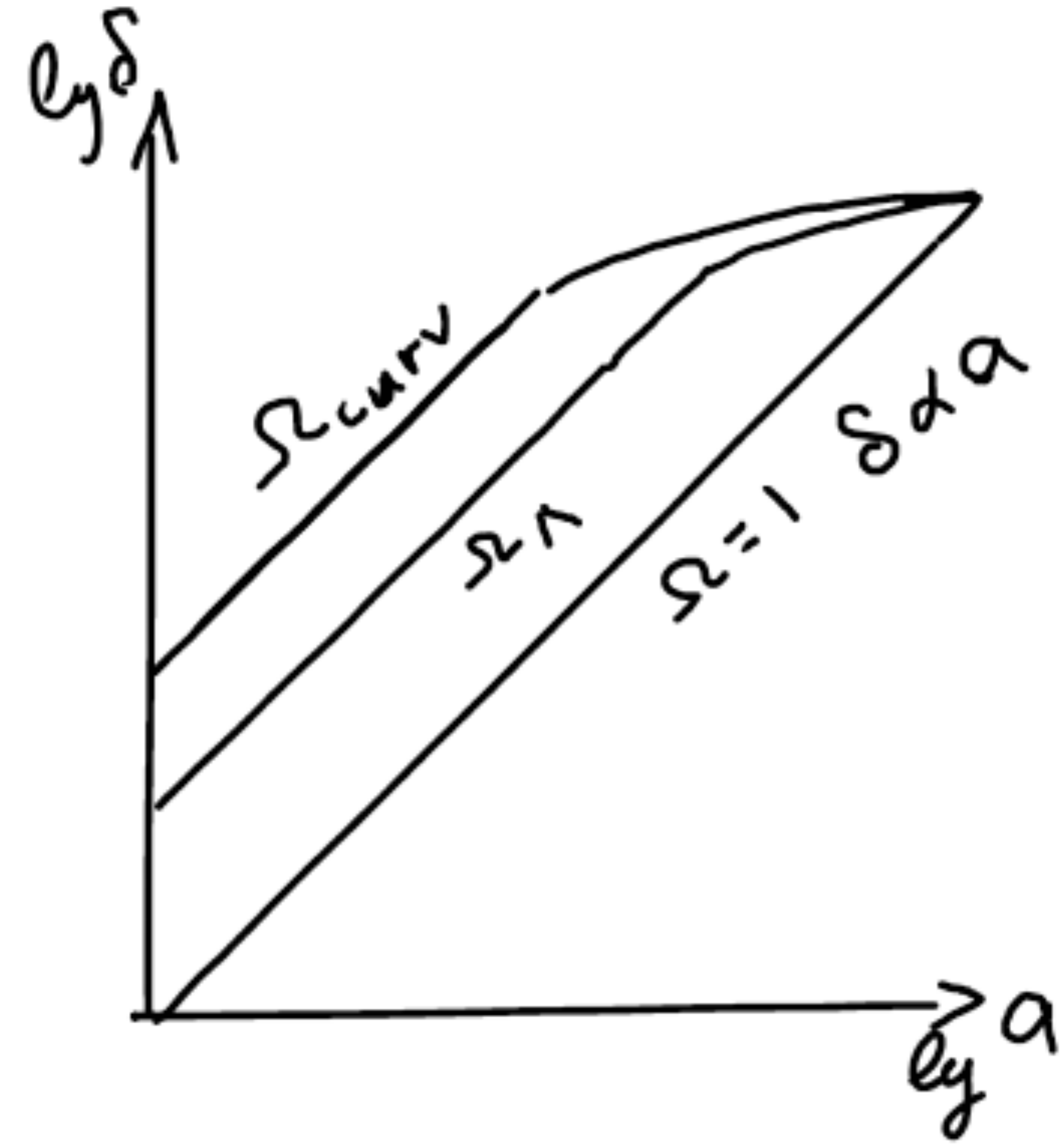






Three models: 1) Flat, matter only
2) Flat +Cosmological constant
3) Open, no cosmological constant

<u>Left panel:</u> the same amplitude of fluctuations at early times

<u>Right panel:</u> The same amplitude at z=0

Case: **waves inside the horizon, relativistic particles dominate**
Growth of perturbations in non-relativistic matter. Fluctuations in the relativistic matter are wiped out by the free streaming.

$$H^2 = \frac{8\pi G}{3}(\rho_m + \rho_\gamma)$$

$$\rho_m \propto a^{-3}, \quad \rho_\gamma \propto a^{-4}$$

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} = 4\pi G \rho_m \delta$$

$$\delta = \frac{\rho_m - \langle \rho_m \rangle}{\langle \rho_m \rangle}$$

Note that delta is the density contrast in matter, not in the total density

Introduce new variable:
Change variable t->y

$$y \equiv \frac{\rho_m}{\rho_\gamma} = \frac{a}{a_{eq}}$$

The equation for the growth rate takes the form:

$$\frac{d^2\delta}{dy^2} + \frac{2+3y}{2y(1+y)}\frac{d\delta}{dy} - \frac{3\delta}{2y(1+y)} = 0$$

The growing solution of this equation can be found by trying: $\ddot{\delta} = 0$
This gives:

$$\delta_{grow} = 1 + \frac{3}{2}y = 1 + \frac{3}{2}\frac{a}{a_{eq}}$$

Transfer Function

Initial spectrum of fluctuations, which was produced during the inflation, is distorted during the evolution of the Universe by different processes.
If δ_k^2 is the square of amplitude of fluctuations at given wavevector $\vec{k}$,

Then we can write: $$\delta_k^2 = A\, k^n\, T(k,t)$$

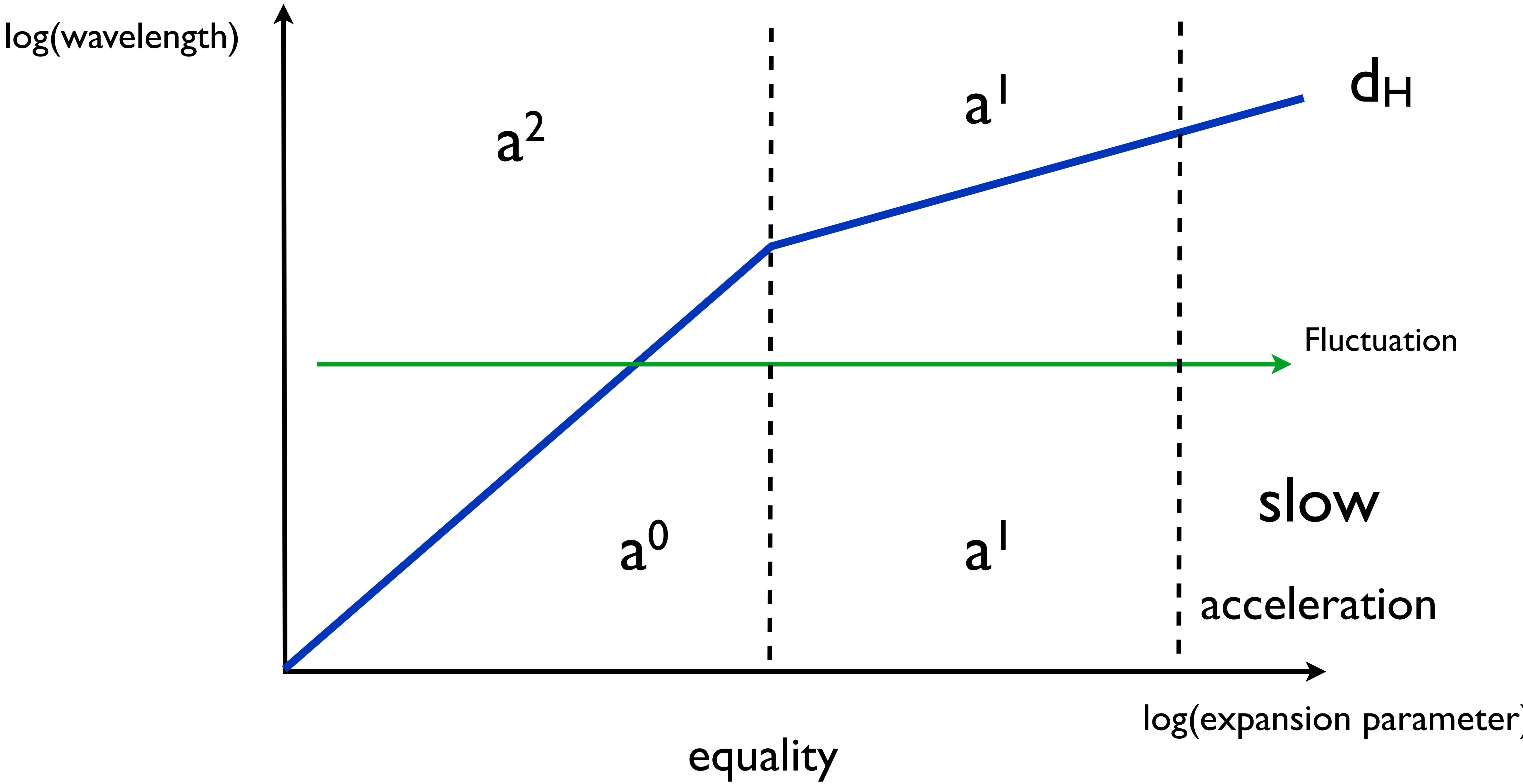
Where A is a normalization constant, k^n is the initial spectrum of perturbations, and
T(k,t) is the transfer function.

One important case: n=1 is called the Harrison-Zeldovich spectrum.

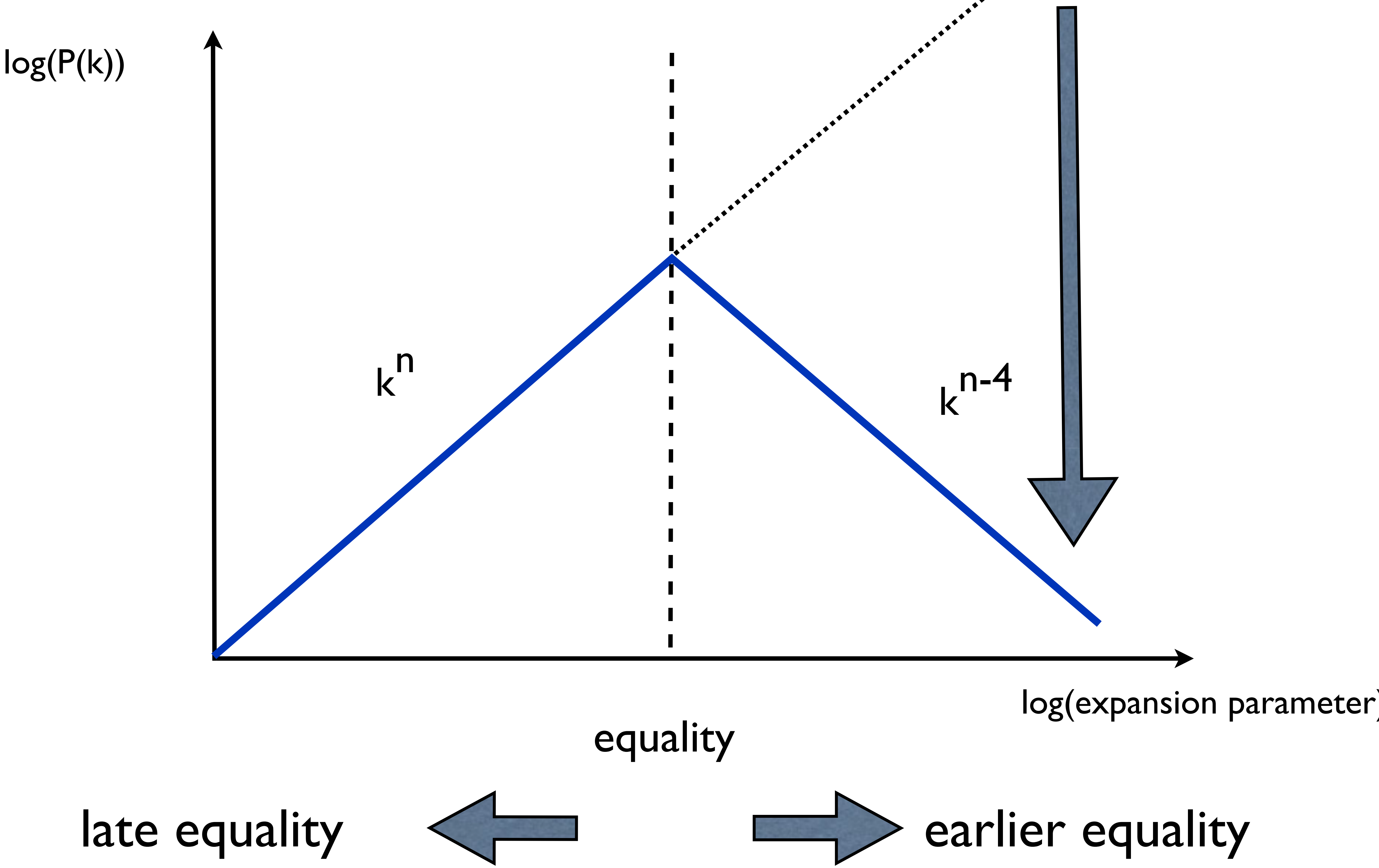
For some cases we can disentangle the dependencies on k and t. For example, for LCDM or open CDM models the shape of the spectrum did not change much after the recombination :

$$T(k,t) = T(k)\, D^2(t)$$

Where D(t) is the growth-factor of fluctuations. Models with hot neutrinos or warm dark matter with some rms velocities of dark matter particles may still experience late changes in growth of perturbations.



Chessboard of growth of adiabatic perturbations



Power spectrum evolution

Spectrum of velocities.

Let's find the relation between perturbations in density and perturbations in peculiar velocity. We are dealing with growing mode, for with there is a unique relation.

$$\begin{cases} \dfrac{\partial \vec{v}}{\partial t} + \dfrac{\dot{a}}{a}\vec{v} = \vec{g} = -\dfrac{\nabla\varphi}{a} \\ \dot{\delta} + \dfrac{1}{a}\nabla\vec{v} = 0 \end{cases}$$

$$\delta(\vec{x}) = \frac{V}{(2\pi)^3}\int d^3k\, \delta_{\vec{k}}\, e^{-i\vec{k}\vec{x}}$$

$$\vec{v}(\vec{x}) = \frac{V}{(2\pi)^3}\int d^3k\, \vec{v}_{\vec{k}}\, e^{-i\vec{k}\vec{x}}$$

Take a plane wave:

$$\vec{v} = \vec{v}_{\vec{k}}\, e^{-i\vec{k}\vec{x}}, \qquad \delta = \delta_{\vec{k}}\, e^{-i\vec{k}\vec{x}}$$

Then: $$\nabla\vec{v} = -i(\vec{k}\cdot\vec{v}_k)\exp(-i\vec{k}\cdot\vec{x})$$

Now the continuity equation can be written in the form:

$$\dot{\delta}_{\vec{k}} + \frac{(-i\vec{k}\vec{v}_k)}{a} = 0$$

For growing mode we have derived the relation: $\vec{v} = a\dfrac{\partial}{\partial t}\left(\dfrac{\vec{g}}{4\pi G\rho a}\right) \Rightarrow$

$$\Rightarrow \vec{v} \parallel \vec{g} \Rightarrow \vec{v} \parallel \vec{k}$$

Velocities

Thus, $$(\vec{K}\vec{v}_K) = K v_K$$

$$v_K = a\,\frac{\dot{\delta}_K}{iK}$$

Write density perturbation in the form:

$$\dot{\delta} = \frac{a}{\delta}\frac{d\delta}{da}\cdot\frac{\dot{a}}{a}\delta = H f(\Omega)\delta$$

$$f(\Omega) \equiv \frac{a}{\delta}\frac{d\delta}{da}$$

We get: $$v_K = \frac{Ha}{iK} f(\Omega)\delta_K \Leftarrow$$ 90degrees rotation relative to δ_K

Power spectrum of velocities is:

$$v_K^2 = H^2 a^2 f^2(\Omega)\,\frac{\delta_K^2}{K^2}$$

Power Spectrum

Consider a filed of density perturbations in a volume V

$$\delta(\vec{x}) = \frac{\rho(\vec{x}) - \rho_b}{\rho_b}$$

The average of the density contrast is equal to zero:

$$\langle \delta(\vec{x}) \rangle = 0$$

Let's find the dispersion of the density contrast:
Decompose the density contrast into the Fourier spectrum

$$\langle \delta^2(\vec{x}) \rangle$$

$$\delta(\vec{x}) = \frac{V}{(2\pi)^3} \int d^3k \, \delta_{\vec{k}} \, e^{-i\vec{k}\vec{x}}$$

$$\delta_{\vec{k}} = \frac{1}{V} \int d^3x \, \delta(\vec{x}) \, e^{i\vec{k}\cdot\vec{x}}$$

Find the dispersion:
$$\langle \delta^2(\vec{x})\rangle = \frac{1}{V}\int d^3x\, \delta^2(\vec{x}) = \frac{1}{V}\int d^3x\, \delta(x)\,\delta(x) =$$

$$= \int \frac{d^3x}{V}\, V\int \frac{d^3K}{(2\pi)^3}\, \delta_{\vec{K}}\, e^{-i\vec{K}\vec{x}}\; V\int\frac{d^3K'}{(2\pi)^2}\, \delta^*_{\vec{K}'}\, e^{i\vec{K}'\vec{x}} = \qquad \delta_{-\vec{K}} = \delta^*_{\vec{K}}$$

$$= \int \frac{d^3x}{V}\, V^2 \int \frac{d^3K\, d^3K'}{(2\pi)^6}\, \delta_{\vec{K}}\, \delta^*_{\vec{K}'}\, e^{-i(\vec{K}-\vec{K}')\vec{x}}$$

Change the order of integration:

$$\langle \delta^2\rangle = \int \frac{d^3K\, d^3K'}{(2\pi)^6}\, \delta_{\vec{K}}\, \delta^*_{\vec{K}'}\; \underbrace{V^2\int \frac{d^3x}{V}\, e^{-i(\vec{K}-\vec{K}')\vec{x}}}_{\delta(\vec{K}-\vec{K}')}$$

$$= V\int \frac{d^3K}{(2\pi)^3}\, |\delta_{\vec{K}}|^2 = \frac{V}{(2\pi)^3}\int_0^\infty 4\pi K^2 dK\, |\delta_K|^2$$

Thus, we get:
$$\langle \delta^2\rangle = \frac{V}{2\pi^2}\int_0^\infty K^2 \delta_K^2\, dK$$

We define the **power spectrum** as

Here the averaging is done over all
Waves with given k and over the whole space

$$P(k) = \langle |\delta_{\vec{k}}|^2 \rangle$$

Correlation function is defined as

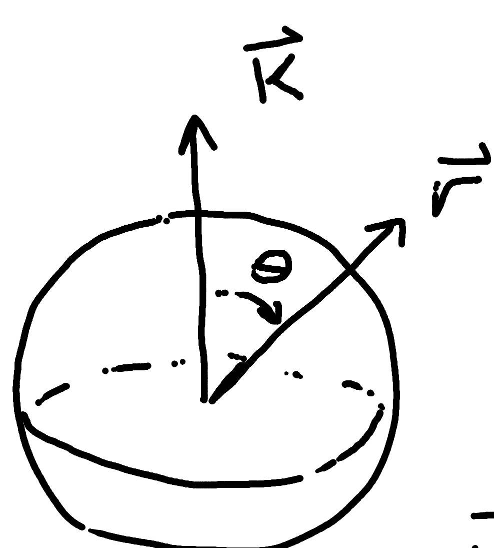
$$\xi(r) = \langle \delta(\vec{x})\,\delta(\vec{x}+\vec{r}) \rangle$$

The averaging is done for the whole
volume and over angles of vector $\vec{r}$

$$\xi(r) = \frac{1}{4\pi}\int d\Omega \int \frac{d^3x}{V} \int \frac{V d^3k}{(2\pi)^3}\, \delta_{\vec{k}}\, e^{-i\vec{k}\vec{x}} \int \frac{V d^3k'}{(2\pi)^3}\, \delta^*_{\vec{k}'}\, e^{i\vec{k}'(\vec{x}+\vec{r})} =$$

$$= \frac{1}{4\pi}\int d(\cos\theta)\, d\varphi \int \frac{V d^3k}{(2\pi)^3}\, \delta_{\vec{k}}\, e^{i\vec{k}\vec{r}}\, \delta^*_{\vec{k}} =$$

$$= \int \frac{V d^3k}{(2\pi)^3}\, \delta_{\vec{k}}\, \delta^*_{\vec{k}}\, \frac{1}{4\pi}\int d(\cos\theta)\, d\varphi\, e^{i\vec{k}\vec{r}}$$



$$e^{i\vec{k}\vec{r}} = \cos(kr\cos\theta) + \cancelto{\text{imaginary}}{i\sin(kr\cos\theta)}$$

$$\frac{1}{4\pi}\int_{-1}^{1} dx \int_0^{2\pi} d\varphi\, \cos(krx) = \frac{\sin(kr)}{kr}$$

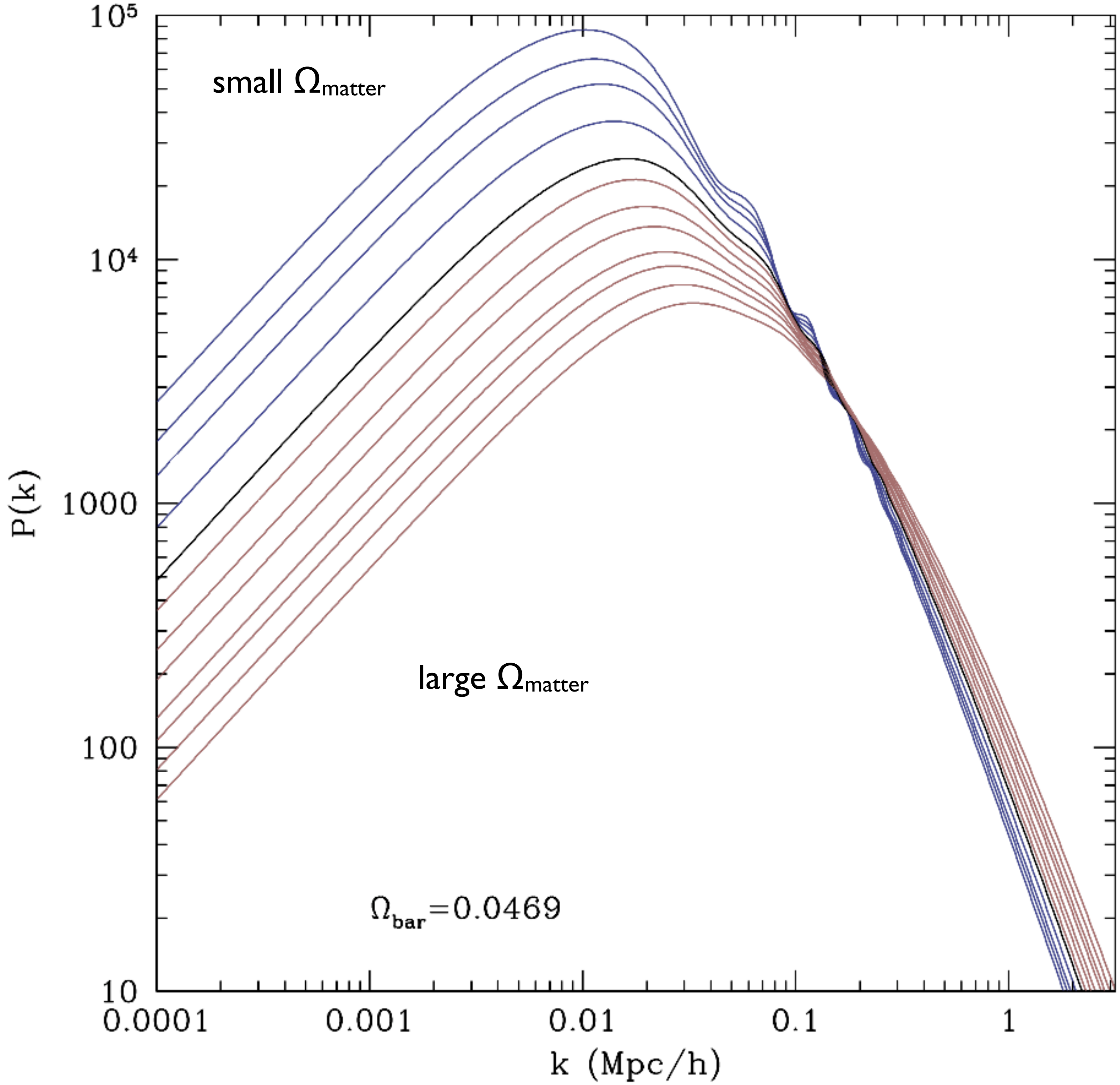
Thus, we get the relation between the correlation function and the power spectrum:

$$\xi(r) = \frac{1}{2\pi^2} \int_0^\infty K^2 dK \, \frac{\sin Kr}{Kr} P(K)$$

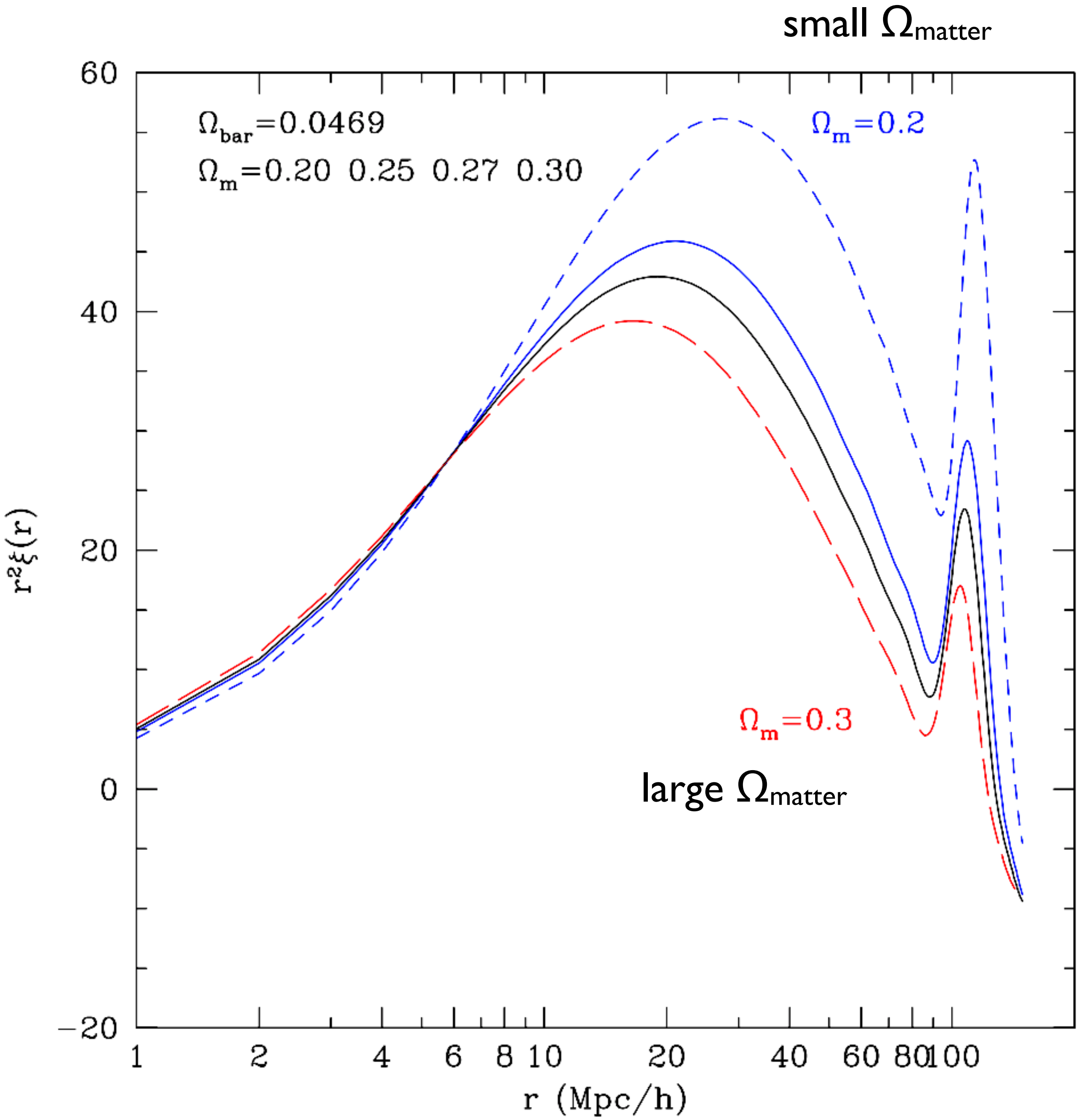
There is an inverse relation:

$$P(K) = 4\pi \int r^2 dr \, \xi(r) \, \frac{\sin Kr}{Kr}$$

Dependance of P(k) on Ω_{matter}
Amplitude of fluctuations and $\Omega_{baryons}$ are fixed.



Dependance of Correlation function on Ω_{matter}
Amplitude of fluctuations is fixed at 5Mpc/h



Dependance of Correlation function on $\Omega_{mbaryon}$
Amplitude of fluctuations is fixed at 5Mpc/h