Perturbations: Non-relativistic matter. Wave shorter than the distance to horizon

Fluid equations in comoving coordinates. We introduce the comoving coordinates:

$$\vec{r} = a(t)\,\vec{x}(t)$$

$$\vec{u} = \frac{d\vec{r}}{dt} = H\vec{r} + \vec{v}$$

$$\vec{v} = a\,\dot{\vec{x}}$$

$\vec{r}$ = proper distance
$\vec{x}$ = comoving distance
$\vec{u}$ = proper velocity
$\vec{v}$ = peculiar velocity

Hydrodynamic equations are written in standard form

$$\left.\frac{\partial \rho}{\partial t}\right|_{\vec{r}} + \nabla_{\vec{r}}(\rho\vec{u}) = 0 \quad \text{continuity equation}$$

$$\left.\frac{\partial \vec{u}}{\partial t}\right|_{\vec{r}} + (\vec{u}\,\nabla_{\vec{r}})\vec{u} = -\frac{1}{\rho}\nabla_{\vec{r}} P - \nabla_{\vec{r}}\Phi \quad \text{Euler equation}$$

$$\nabla^2_{\vec{r}}\Phi = 4\pi G\rho \quad \text{Poisson equation}$$

Introduce peculiar gravitational potential:

$$\Phi = \underbrace{\frac{2}{3}\pi G\rho_b r^2}_{\text{potential of the homogeneous background}} + \varphi$$

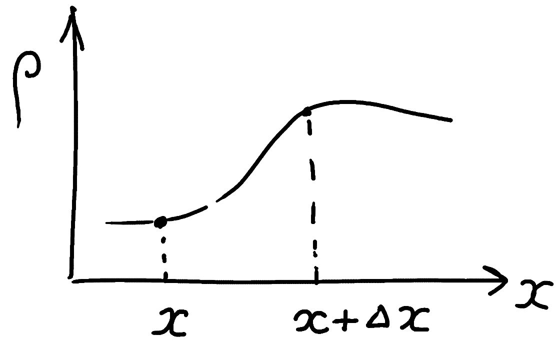
peculiar potential

Now we need to change variables from proper to comoving. Spatial derivatives are simple to handle:

$$\frac{\partial}{\partial r} = \frac{1}{a}\frac{\partial}{\partial x}$$

Time derivatives at constant $\vec{r}$ should be changed to time derivates at constant $\vec{x}$
This gives two terms: derivative with respect to time at constant comoving coordinate and a term due to the fact that constant r means that x changes. So, we get physical property at different comoving coordinates: a gradient should be present:

Here is an example: density



In Δt time interval density changes by:

$$\Delta\rho\Big|_r = \Delta\rho\Big|_x + \frac{\Delta\rho}{\Delta x}\,\Delta x$$

Here Δx is defined by condition:

$$\vec{r} = const \Rightarrow \Delta a \cdot x + a\Delta x \Rightarrow \Delta x = -\frac{\Delta a}{a}\,x$$

Now: $\Delta \to d$

$$\frac{\partial\rho}{\partial t}\Big|_r = \frac{\partial\rho}{\partial t}\Big|_x - \frac{da}{a\,dt}\left(\vec{x}\,\nabla_{\vec{x}}\right)\rho$$

In the same manner we get:

$$\frac{\partial\vec{u}}{\partial t}\Big|_{\vec{r}} = \frac{\partial\vec{u}}{\partial t}\Big|_{\vec{x}} - \frac{1}{a}\frac{da}{dt}\left(\vec{x}\,\nabla_{\vec{x}}\right)\vec{u}$$

Because we had: $\vec{u} = \dot{a}\vec{x} + \vec{v} \Rightarrow \dfrac{\partial\vec{u}}{\partial t}\Big|_x = \ddot{a}\vec{x} + \dfrac{\partial\vec{v}}{\partial t}$

Now we need to change variables from $\vec{u}$ to $\vec{v}$

$$\nabla_r(\rho\vec{u}) = \frac{1}{a}\nabla_x\left(\rho\dot{a}\vec{x} + \rho\vec{v}\right) = \frac{1}{a}\nabla_x(\rho\vec{v}) + \frac{\dot{a}}{a}\nabla_x(\rho\vec{x}) =$$

$$= \frac{1}{a}\nabla_x(\rho\vec{v}) + \frac{\dot{a}}{a}\vec{x}\nabla_x\rho + \frac{\dot{a}}{a}\rho\nabla\vec{x} =$$

$$\frac{1}{a}\nabla_x(\rho\vec{v}) + \frac{\dot{a}}{a}\vec{x}\nabla_x\rho + 3\frac{\dot{a}}{a}\rho$$

$\nabla\vec{x} = 3$

$\nabla_r^2 r^2 = 6$

$\nabla_r r^2 = 2\vec{r}$

$$\vec{u}\cdot\nabla_r\vec{u} = \frac{\dot{a}}{a}\vec{x}\nabla_x\vec{u} + \frac{\vec{v}}{a}\nabla_x\left(\dot{a}\vec{x} + \vec{v}\right) =$$

$$= \frac{\dot{a}}{a}\vec{x}\nabla_x\vec{u} + \frac{\dot{a}}{a}\vec{v} + \frac{\vec{v}}{a}\nabla_x\vec{v}$$

Put all the terms in the equations of hydrodynamics and omitting indexes r in derivatives:

$$\frac{\partial\rho}{\partial t} + 3\frac{\dot{a}}{a}\rho + \frac{1}{a}\nabla(\rho\vec{v}) = 0$$

$$\frac{\partial \vec{v}}{\partial t} + \frac{\dot{a}}{a}\vec{v} + \frac{1}{a}(\vec{v}\nabla)\vec{v} = -\frac{1}{a}\frac{\nabla p}{\rho} - \frac{1}{a}\nabla\varphi - \underbrace{-\ddot{a}\vec{x} - \frac{1}{a}\nabla_x\left(\frac{2\pi}{3}G\rho_b r^2\right)}_{-\left(\ddot{a} + \frac{4\pi}{3}G\rho_b a\right)\vec{x} = 0}$$

Thus, the Euler equation gives:

$$\frac{\partial \vec{v}}{\partial t} + \frac{\dot{a}}{a}\vec{v} + \frac{1}{a}(\vec{v}\nabla)\vec{v} = -\frac{1}{a}\frac{\nabla p}{\rho} - \frac{1}{a}\nabla\varphi$$

Now, let's handle the Poisson equation:

$$\nabla_r^2 \Phi = \frac{1}{a^2}\nabla_x^2\varphi + \nabla_r^2\left(\frac{2\pi}{3}G\rho_b r^2\right) = \frac{1}{a^2}\nabla_x^2\varphi + 4\pi G\rho_b$$

Thus, the Poisson equation takes the form:

$$\nabla^2\varphi = 4\pi G a^2(\rho - \rho_b)$$

Note that gravity effectively gets "stronger" with time because there is term a^2.
Only deviations from the homogeneity enter the right-hand-side ("source term")
If local density is smaller than the average density, perturbations in grav. Potential are negative. This acts as a negative mass.

Using the equations, we can re-cast them for equations of individual free particles:

$$\frac{d\vec{v}}{dt} + \frac{\dot{a}}{a}\vec{v} = -\frac{1}{a}\nabla\varphi$$

Introduce the momentum: $\vec{p} \equiv a\vec{v} = a^2\dot{\vec{x}} \Rightarrow \frac{d\vec{p}}{dt} = -\nabla\varphi$

Regime of small perturbations: $\rho(\vec{x},t) \equiv \rho_b(t)\left(1+\delta(\vec{x},t)\right)$

$\delta \ll 1$

We can neglect terms of higher order: $\delta^2, v^2, \ldots$

Continuity equation:

$$(1+\delta)\frac{\partial \rho_b}{\partial t} + \rho_b \frac{\partial \delta}{\partial t} + \frac{3\dot{a}}{a}(1+\delta)\rho_b + \frac{\rho_b}{a}\nabla\vec{v} = 0$$

This gives two equations:

$$\frac{\partial \rho_b}{\partial t} + 3\frac{\dot{a}}{a}\rho_b = 0$$

$$\dot{\delta} + \frac{1}{a}\nabla\vec{v} = 0$$

Euler equation:

$$\frac{\partial \vec{v}}{\partial t} + \frac{\dot{a}}{a}\vec{v} = -\frac{\nabla P}{a\rho_b} - \frac{1}{a}\nabla\varphi$$

Poisson equation:

$$\nabla^2\varphi = 4\pi G a^2 \rho_b \delta$$

Take divergence of Euler eq:

$$\frac{\partial \nabla v}{\partial t} + \frac{\dot{a}}{a}\nabla\vec{v} = -\frac{\nabla^2 P}{a\rho_b} - 4\pi G\rho_b a\delta$$

From continuity eq we get: $\nabla\vec{v} = -a\dot{\delta}$

Combining these equations (get rid of div(v) and grav. Potential):

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} = \frac{\nabla^2 P}{a^2\rho_b} + 4\pi G\rho_b\delta$$

If P is negligable, we have a linear differential equation, which does not depend on the wavelength of the perturbation. In other words, in this regime all perturbations grow with the same rate.

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} = 4\pi G\rho_b\delta, \qquad P \simeq 0$$

Peculiar velocity and peculiar acceleration:

Equations:

$$\nabla^2 \varphi = 4\pi G a^2 (\rho - \rho_b) \Rightarrow \varphi(\vec{x}) = -Ga^2 \int d^3x' \, \frac{\rho(x') - \rho_b}{|x' - x|}$$

$$\vec{g}(\vec{x}) = -\frac{\nabla \varphi}{a} = Ga \int d^3x' (\rho - \rho_b) \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3}$$

Peculiar acceleration can be re-written in slightly different form:

$$\vec{g}(\vec{x}) = G a \rho_b \int d^3x' \; \delta(\vec{x}, t) \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3}$$

Relation between g and v in the linear regime:

$$\nabla \vec{g} = -4\pi G \rho_b a \delta \Rightarrow \delta = -\frac{\nabla \vec{g}}{4\pi G \rho_b a} \Rightarrow \dot{\delta} = -\frac{\partial}{\partial t}\left(\frac{\nabla \vec{g}}{4\pi G \rho_b a}\right)$$

From continuity equation we get: $\dot{\delta} = -\frac{\nabla \vec{v}}{a}$

Thus: $$\vec{v} = a \frac{\partial}{\partial t}\left(\frac{\vec{g}}{4\pi G \rho_b a}\right) + \frac{\vec{F}(\vec{x})}{a} \quad \Big| \quad \nabla \vec{F} = 0$$

decaying mode

For growing mode we find that velocity and acceleration are related:

$$\vec{v} = a \frac{\partial}{\partial t}\left(\frac{\vec{g}}{4\pi G \rho_b a}\right) \Rightarrow \vec{v} \parallel \vec{g}$$

$$\vec{g} = G \rho_b a \int d^3x' \, \delta(x') \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \Rightarrow \left(\frac{\vec{g}}{G\rho_b a}\right)^{\cdot} = \int (\dots \dot{\delta}) = \frac{\dot{\delta}}{\delta} \int \dots$$

acceleration

Tuesday, September 18, 2007
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Thus, we get for v-g relation:

$$\vec{v} = \frac{\vec{g}}{4\pi G\rho_b} \frac{1}{\delta} \frac{d\delta}{dt}$$

$$\left(\frac{\vec{g}}{G\rho_b a} \right)^{\cdot} = \frac{\dot{\delta}}{\delta} \frac{\vec{g}}{G\rho_b a}$$

Re-write the derivative of density contrast:

$$\frac{1}{\delta} \frac{d\delta}{dt} = \frac{a}{\delta} \frac{d\delta}{da} \frac{da}{a\,dt} \equiv f(\Omega) H$$

$$f(\Omega) \simeq \Omega^{0.6}, \quad \Omega = \frac{8\pi G}{3} \frac{\rho_b}{H^2}$$

omega matter

Finally we write:

$$\vec{v} = \frac{2}{3} \frac{f(\Omega)}{H\Omega} \vec{g}$$

General equation for growth of perturbations is:

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} = 4\pi G\rho_b\delta + \frac{\nabla^2 P}{\rho_b a^2}$$

Particular cases. The simplest case is ***the flat Universe, wavelength shorter than the horizon, waves longer than the Jeans mass.*** This is also the case of the cold dark matter (negligible random velocities):

$$P=0,\ \Omega_m=1,\ \rho_b=\frac{1}{6\pi G t^2}\propto a^{-3}$$

Friedmann equation is: $$\left(\frac{\dot{a}}{a}\right)^2=\frac{8\pi G}{3}\rho=\left(\frac{2}{3t}\right)^2$$

Now the equation for the density contrast can be written as:

$$\ddot{\delta}+\frac{4}{3}\frac{\dot{\delta}}{t}=\frac{2}{3}\frac{\delta}{t^2}$$

Solution of this equation is found in the form $\delta = At^n$

This gives: $$n(n-1)+\frac{4}{3}n=\frac{2}{3} \Rightarrow n=\frac{2}{3} \text{ or } n=-1$$

Thus, the general solution is:

$$\delta(x,t)=\underbrace{A(x)t^{2/3}}_{\text{growing}}+\underbrace{B(x)t^{-1}}_{\text{decaying}}$$

Because in this case $a\propto t^{2/3}$, we can re-write the growing mode in more elegant form:

$$\delta_{grow}(x,t)=\delta_{init}(x)\frac{a}{a_{init}} \Rightarrow \delta\propto a$$

Velocity for this mode is found from the continuity equation:

$$\nabla\vec{v}=-a\dot{\delta}$$

Assuming the form for the density perturbation is $\delta=A(\vec{x})t^{2/3}$,

How do we get velocity: $\vec{v}=c(t)\vec{f}(\vec{x})$?

$$\nabla\vec{v}=-a\frac{\partial\delta}{\partial t} \Rightarrow c(t)\nabla\vec{f}=-a(t)A(\vec{x})\frac{2}{3}t^{-1/3}$$

From this we can easily find the time dependence: $c(t)\propto t^{1/3}\propto\sqrt{a}$

$$\vec{v}_{grow} = \sqrt{\frac{a}{a_{init}}}\,\vec{v}_{init}(\vec{x})$$

Now, normalize the velocity:
Note that v_{init} and δ_{init} are not independent.
We will find their relation later using spectral formalism.

Another case: Open Universe, only non-relativistic matter $\quad p=0$

$$H^2 = \frac{8\pi G \rho_b}{3} - \frac{K}{a^2}, \quad \text{here } K = -H_0^2(1-\Omega_0)$$

$$\rho_b = \frac{3H_0^2}{8\pi G}\frac{\Omega_0}{a^3} \rightarrow \Omega(t) = \frac{H_0^2}{H^2}\frac{\Omega_0}{a^3}$$

Introduce new variable: $x \equiv \frac{1-\Omega(t)}{\Omega(t)} = \frac{1-\Omega_0}{\Omega_0}a(t)$

Now change the time variable in the equation for density contrast:

$$\frac{d^2\delta}{dx^2} + \frac{3+4x}{2x(1+x)}\frac{d\delta}{dx} - \frac{3\delta}{2x^2(1+x)} = 0$$

The growing solution of this simple equation is just wonderful:

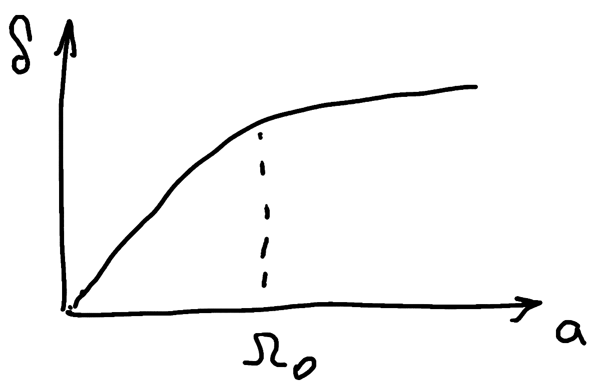
$$\delta = 1 + \frac{3}{x} + \frac{3(1+x)^{1/2}}{x^{3/2}}\ln\left[(1+x)^{1/2} - x^{1/2}\right]$$

Asymptotic behavior of this solution $\quad x \ll 1 \rightarrow \delta \propto t^{2/3}$

$$x \gg 1 \rightarrow \delta \sim 1$$

Fluctuations grow untill $x \simeq 1$,
Which corresponds to $a \approx \Omega_0$

Then they stop growing



Case: ***flat Universe with cosmological constant***

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda c^2}{3}$$

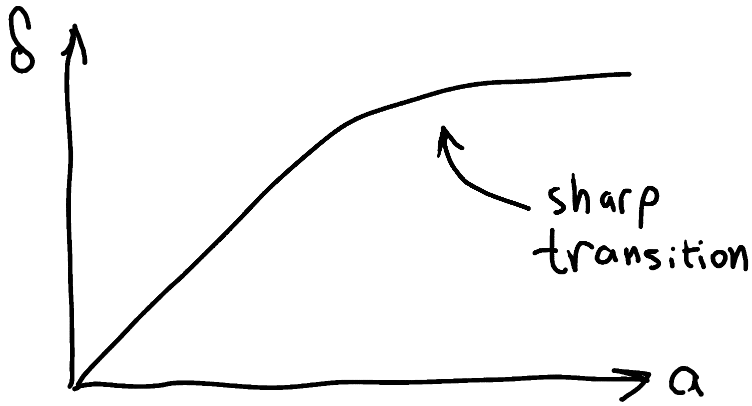
Solution for the growth of small perturbations is

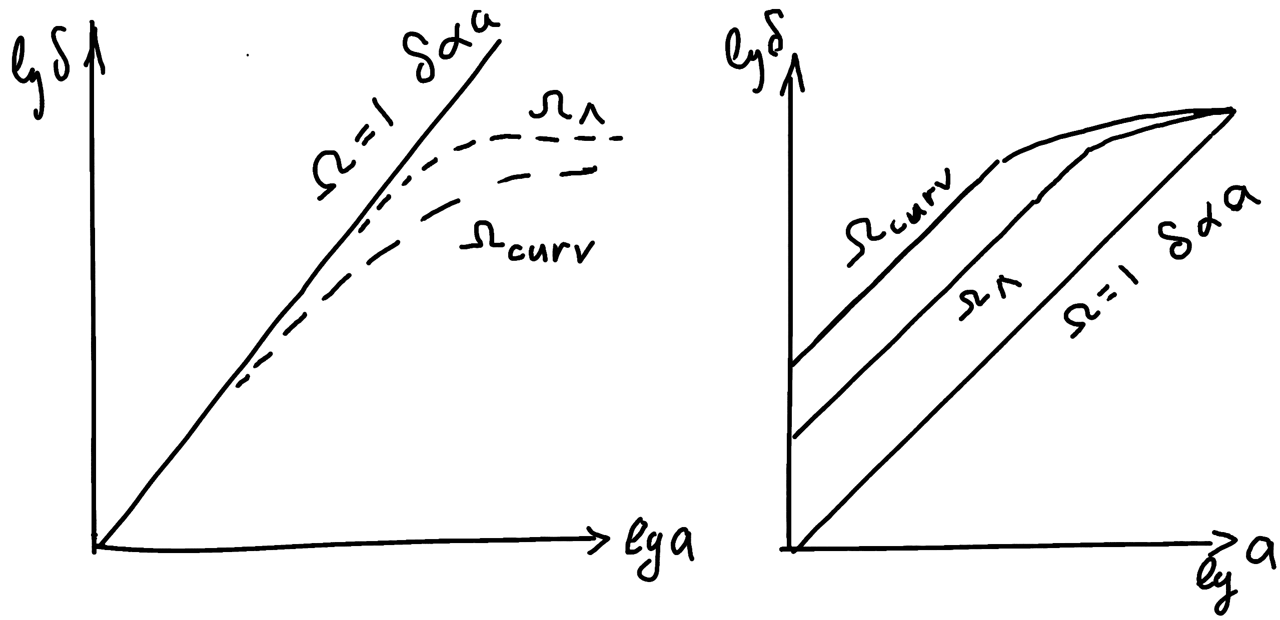
$$\delta = \frac{1}{x_0}\frac{\sqrt{1+x^3}}{x^{3/2}}\int_0^x \frac{x^{3/2}\,dx}{[1+x^3]^{3/2}}$$

$$\Omega_\Lambda = \Omega_{\Lambda,0}\frac{H_0^2}{H^2}$$

$$x = x_0 a = \left(\frac{\Omega_{\Lambda,0}}{\Omega_0}\right)^{1/3} a$$

$$x_0 = \left(\frac{\Omega_{\Lambda,0}}{\Omega_0}\right)^{1/3}$$





Three models: 1) Flat, matter only
2) Flat +Cosmological constant
3) Open, no cosmological constant

<u>Left panel:</u> the same amplitude of fluctuations at early times

<u>Right panel:</u> The same amplitude at z=0

Case: **waves inside the horizon, relativistic particles dominate**
Growth of perturbations in non-relativistic matter. Fluctuations in the relativistic matter are wiped out by the free streaming.

$$H^2 = \frac{8\pi G}{3}(\rho_m + \rho_\gamma)$$

$$\rho_m \propto a^{-3}, \quad \rho_\gamma \propto a^{-4}$$

$$\delta = \frac{\rho_m - \langle \rho_m \rangle}{\langle \rho_m \rangle}$$

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} = 4\pi G \rho_m \delta$$

Note that delta is the density contrast in matter, not in the total density

Introduce new variable:
Change variable t->y

$$y \equiv \frac{\rho_m}{\rho_\gamma} = \frac{a}{a_{eq}}$$

The equation for the growth rate takes the form:

$$\frac{d^2\delta}{dy^2} + \frac{2+3y}{2y(1+y)}\frac{d\delta}{dy} - \frac{3\delta}{2y(1+y)} = 0$$

The growing solution of this equation can be found by trying: $\ddot{\delta} = 0$
This gives:

$$\delta_{grow} = 1 + \frac{3}{2}y = 1 + \frac{3}{2}\frac{a}{a_{eq}}$$

Note before the equality the fluctuations grow very little.
After the equality $\delta \propto a$

Situation: fluctuations in baryons, waves inside the horizon

We deal with ideal fluid with pressure P. For non-expanding medium the Jeans length was found by equating the time needed for a wave to travel across an object to object's free fall time. Better way: find critical regime for dispersion relation for homogeneous gas with given density and temperature:

$$l_J \approx \frac{v_s}{\sqrt{G\rho}}$$

For ideal fluid velocity of the sound is given by:

$$v_s^2 = \frac{dP}{d\rho} = \gamma \frac{P}{\rho} = \frac{\gamma}{\mu m_H} kT$$

$$P = nkT \qquad \rho = \mu n m_H$$

For gas before the recombination the pressure is by far provided by radiation $P = P_{gas} + P_\gamma \simeq P_\gamma = \frac{\rho_\gamma c^2}{3}$

When pressure waves moves through the gas it creates variations in density and temperature of the radiation:

$$\Delta\rho_{gas} \simeq \Delta\rho_\gamma$$

As the result, sound velocity is $v_s = \frac{c}{\sqrt{3}}$

After recombination, photon move freely and gas pressure drops to normal P= nkT. This gives sound speed of few km/s.

Growth of fluctuations in gas: