Kinematics of particles in an expanding Universe

Expansion parameter: for freely moving particles and waves all scales change with time proportionally to some universal scaling factor, which we call *expansion parameter.* It is convenient to normalize the expansion parameter in such a way that:

$a = 1$ at present and $a = 0$ at $t = 0$

IF a is normalized in this way, then $a = \frac{1}{1+z}$ where z Is the redshift. For example, if l_0 is a distance between two galaxies at present, then at moment t the distance between then is

$$l(t) = l_0 a(t)$$

Note, that this is valid only for objects, which participate in the global expansion and are not gravitationally bound (or interact) to each other.

Comoving and Proper coordinates: We introduce proper distances as physical distances between objects at some particular time t or expansion parameter a(t). Comoving distances are coordinate distances. If at some expansion parameter a the proper distance was l, then we can extrapolate it to the present moment assuming that the object expands as the whole Universe: $x = l/a$.
Relation between proper coordinates $\vec{r}$ and comoving coordinates $\vec{x}$ are:

$$\vec{r} = a(t)\vec{x} \qquad (*)$$

In general, comoving coordinates may change with time. This happens when perturbations are present and the Universe is not exactly homogeneous.

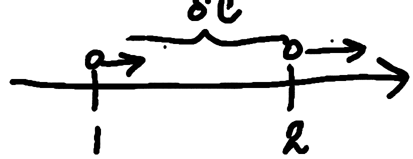
Differentiate eq($*$) with time:

$$\dot{\vec{r}} = \vec{v} = \dot{a}\vec{x} + a\dot{\vec{x}} = \frac{\dot{a}}{a}(a\vec{x}) + a\dot{\vec{x}} =$$
$$= H\vec{r} + \vec{v}_{pec}, \quad H = \frac{\dot{a}}{a}, \quad \vec{v}_{pec} = a\dot{\vec{x}}$$

Here H is the Hubble constant and v_{pec} is the peculiar velocity - deviation from the perfect Hubble flow.

Light: Consider two observers at proper separation $\delta l = a(t)\delta x$ The difference in velocity between the observers is:

$$\delta v = \frac{\dot{a}}{a}\delta l = H\delta l$$



The first observer sees that the second observer moves away from him. Thus, the Doppler shift of light at the position of the second observer is:

$$d\nu = -\nu \frac{\delta v}{c} = -\nu \frac{H\delta l}{c}$$

Light will reach the second observer in time $\delta t = \frac{\delta l}{c}$
Thus,

$$\frac{d\nu}{\nu} = \frac{\delta a}{a\,\delta t} \cdot \frac{\delta l}{c} = -\frac{\delta a}{a} \Rightarrow \boxed{\nu = \nu_0 \frac{a_0}{a}}$$

We introduce the redshift Z as:

$$\frac{\nu_{em}}{\nu_{obs}} \equiv 1 + z = \frac{1}{a}$$

Here we explicitly used condition $a_0 = 1$
Now, we make the same derivation, but for more general case: motion of any free particle. The particle has a peculiar velocity v at the position of the first observer and peculiar velocity v' at the position of the second observer. The second observer moves with relative velocity

$$\delta v = \frac{\dot{a}}{a}\delta l = \frac{\dot{a}}{a} v\delta t = v\frac{\delta a}{a}$$

The relative velocity of the particle at the second observer is:

$$v' = \frac{v - \delta v}{1 - \frac{v\delta v}{c^2}} \approx v - \delta v\left(1 - \frac{v^2}{c^2}\right)$$

The change in the peculiar velocity is $dv = v' - v = -\delta v\left(1 - \frac{v^2}{c^2}\right) =$

$$= -v\frac{\delta a}{a}\left(1 - \frac{v^2}{c^2}\right)$$

We can write this as:

$$\frac{dv}{v\left(1 - \frac{v^2}{c^2}\right)} = -\frac{\delta a}{a} \quad (*)$$

Introduce momentum of a particle: $$p \equiv \frac{v}{\sqrt{1 - \frac{v^2}{c^2}}}$$

The eq(*) is a full derivative: dp/p. Thus, the equation can be integrated over time:

$$p = p_0 / a$$

For non-relativistic particles $v \ll c \Rightarrow p = v$ $\qquad v_{pec} \propto \frac{1}{a}$

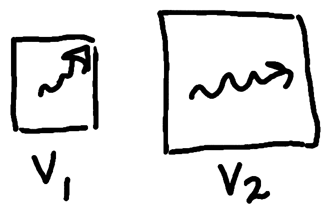
Conclusion: peculiar velocities, which are not supported by perturbations in gravity, decay as the Universe evolves.
For relativistic particles: $E = cp = h\nu \propto 1/a$

Changes in the temperature of radiation: use the first law of thermodynamics:

$$dE = TdS - pdV, \quad dS = 0$$

$$E = \rho c^2 V, \quad \rho = \frac{\sigma T^4}{c^2},$$

$$p = \frac{\rho c^2}{3}$$



Thus,

$$\frac{d\rho}{\rho} = -\frac{4}{3}\frac{dV}{V} = -4\frac{da}{a}$$

$$\Rightarrow \rho_r \propto a^{-4} \qquad T = \frac{T_0}{a} = T_0(1+z)$$

Conclusions:

- The contribution of relativistic particles (e.g. photons) declines with the expansion parameter faster than the contribution of non-relativistic particles. Thus, in the past relativistic particles such as photons and neutrinos played much more important role than they do today.
- The temperature of radiation at given redshift does *not* depend on how the Universe was expanding and what it is made of. Lamda or no lambda, curvature or no curvature - this all does not affect the temperature. (It does affect the mapping from the redshift to real time, though). An important assumption is that the number of photons in a comoving volume is preserved. This may not be true at every stage of the evolution of the Universe.