Geometry of curved space

Special Relativity: line element

Events in space-time are described by their 4-coordinates:

$$ct, x, y, z : x^\alpha, \alpha = 0,1,2,3$$

In special relativity a preserved quantity (under rotation and Lorentz transformation) is called line element. If there are two events, separated by dt, dx, dy, dz than the line element is

$$ds^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2)$$

If $ds^2 > 0$ then we can chose the frame in such a way that $dx = dy = dz = 0$. Then ds is just <u>proper time.</u>

If $ds^2 < 0$ we can chose the frame in such a way that $dt = 0$. Then $\sqrt{-ds^2}$ is proper distance.

For light traveling from one place to another: $ds = 0$

In spherical system of coordinates:

$$ds^2 = c^2 dt^2 - \left[dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2) \right] = c^2 dt^2 - \left[dr^2 + r^2 d\Omega^2 \right]$$

In general relativty the line element is also preserved, but the relation between coordinates and ds is more general:

$$ds^2 = g_{\alpha\beta} \, dx^\alpha dx^\beta \quad \leftarrow \text{summation over repeating indexes.}$$

Tensor $g_{\alpha\beta}$, $\alpha,\beta = 0,1,2,3$ is called metric. It is a set of functions of coordinates. It is a symmetric tensor: $g_{\alpha\beta} = g_{\beta\alpha}$. There are 10 independent functions in $g_{\alpha\beta}$

Equivalence principle: Within a small volume element V we always can find a non-inertial frame such that metric $g_{\alpha\beta}$ has the shape of the Minkovski metric:

$$g_{\alpha\beta} = \eta_{\alpha\beta}, \text{ where } \eta_{\alpha\beta} = \begin{bmatrix} 1 & & & \\ & -1 & & 0 \\ & 0 & -1 & \\ & & & -1 \end{bmatrix} \text{ and } \frac{\partial g_{\alpha\beta}}{\partial x'^\gamma} = 0$$

Isotropy of the space implies that $g_{0i} = 0$ $(i = 1,2,3)$ Otherwise we would had a vector $v_i = g_{0i}$, which would have defined a preferential axis for the whole Universe.

It is important to introduce "fundamental observers" (a frame) for which the Universe is isotropic and homogeneous. This implies that

$$g_{00} = 1$$

Thus, we narrowed down out choice of the metric to the form:

$$ds^2 = c^2 dt^2 - G_{ij} dx^i dx^j$$

The most general form of metric, which is homogeneous and isotropic is the Friedmann-Robertson-Walker metric:

$$ds^2 = c^2 dt^2 - a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right]$$

Here $a(t)$ is the radius of the Universe and coordinate $r = (0-1)$
Note, that we always can move dimensions from r to a.
It does not change the shape of the metric, but sometimes it is more convenient to a as an expansion factor (dimensionless quantity from 0 to 1). If r is dimensionless, then $k = -1, 0, +1$

Let's consider the spatial part of the metric: $d\ell^2 = a^2 \left[\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2) \right]$

For k=0 this is just a square of the distance between two events $\{ a dr, d\theta, d\varphi \}$

For k=1 we embed a 3d sphere of radius R into an abstract 4d space. Analog of this procedure is a 2d sphere in 3d space:
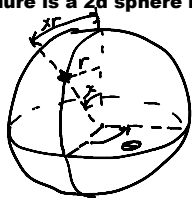
Distances between two points on the 3d-sphere are:

$$d\ell^2 = dx^2 + dy^2 + dz^2 + dw^2 \quad \text{4th coordinate}$$

We also need to use the condition that the points are on the 3d-sphere: $R^2 = r^2 + w^2$

Using this condition we eliminate the 4th coordinate:

$$w\,dw = -r\,dr \implies dw = -\frac{r\,dr}{\sqrt{R^2 - r^2}}$$

Thus,

$$d\ell^2 = dr^2 + r^2 d\Omega^2 + \frac{r^2 dr^2}{R^2 - r^2} = \frac{R^2}{R^2 - r^2} dr^2 + r^2 d\Omega^2$$

$$\implies d\ell^2 = \frac{dr^2}{1 - \frac{r^2}{R^2}} + r^2 d\Omega^2 \quad \text{This is the same as in FRW if we assume } R = 1 \text{ and } r = 0 - 1$$

The length of a circle is $2\pi r$ **and area of a sphere is** $4\pi r^2$

So, it looks as if this a flat geometry. But for flat geometry radius is physical radius. Here radius
is a coordinate distance. Physical distance ℓ corresponding to coordinate r is:

$$\ell = \chi R = R \int_0^r \frac{dr}{\sqrt{1 - r^2/R^2}} = R \arcsin\left(\frac{r}{R}\right)$$

Thus, χ is the measure of physical distance. Now, the relation between proper distance
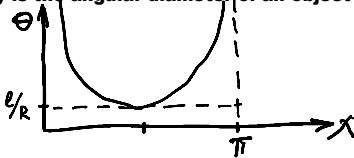and area of a sphere is different from flat geometry:

$$S = 4\pi R^2 \sin^2 \chi$$

The total volume of the universe is:

$$V_{tot} = R^3 \int_0^\pi d\chi \int d\theta \int d\varphi \sin^2\chi \sin\theta \qquad = 2\pi^2 R^3$$

**Another interesting quantity is the angular diameter of an object of given constant linear
size**

$$\Theta = \frac{d}{R \sin \chi}$$

Case k=-1: negative curvature. We make a substitution in previous derivations: $R \Rightarrow iR$
This is what we get: In this case $\chi > 0$. Total volume is infinite.

Surface area is

$$S = 4\pi R^2 \sinh^2 \chi$$

$$d\ell^2 = \frac{d r^2}{1 + \frac{r^2}{R^2}} + r^2 d\Omega^2$$

$$\chi = \int \frac{dr}{\sqrt{1 - kr^2}} = \operatorname{arcsinh}\left(r/R\right)$$

FRW metric can be re-written in a form, which is valid for all cases of k:

$$ds^2 = c^2 dt^2 - a^2(t)\left[d\chi^2 + f^2(\chi)\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right]$$

Where

$$\chi = \int \frac{dr}{\sqrt{1 - kr^2}} = \begin{cases} \arcsin r, & k = 1 \\ r, & k = 0 \\ \operatorname{arcsinh} r, & k = -1 \end{cases} \qquad f(\chi) = \begin{cases} \sin \chi \\ \chi \\ \sin \chi \end{cases}$$

Some useful facts:

- **Phase space density does not change as the Universe expands** $f(\vec{x}, \vec{p}, t)$ $dN = f\,d^3x\,d^3p$
- **Surface brightness at a given frequency changes as:** $F_{em}(\nu(1+z)) = F_{obs}(\nu)(1+z)^3$
- **Total surface brightness (integrated over all frequencies changes as** $(1+z)^4$
- **Flux for given luminosity scales as** $\varepsilon = \dfrac{L}{4\pi R^2 (1+z)^4}$

The last two scaling laws are easy to understand $T \propto (1+z)$
if we recollect that temperature of radiation
decreases as inverse of expansion parameter . $I = \int_0^\infty F(\nu, T)\, d\nu = \text{const} \cdot T^4$
For radiation the total energy per unit solid angle
is an integral of flux over all frequencies:
Thus, the larger is the redshift, the smaller is the
flux, which we receive at z=0.

Distance to the horizon   Question: what fraction of the Universe can be possibly in causal contact? We need to find the proper distance at z=0 for a point, from which we receive light for the first time. This will be the distance to the horizon.

We chose a frame, which is most convenient for integration. We are at the origin and the point, from which we receive the light is along the radius.

We start with FRW:

$$ds^2 = c^2 dt^2 - a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2 d\Omega^2\right]$$

Fix time and find proper distance to an object with coordinate distance r_H

$$d_H(t) = a(t)\int_0^{r_H} \frac{dr}{\sqrt{1-\kappa r^2}}$$

In order to take the integral, we need to know the coordinate distance to the point, from which we receive the light for the first time in the history of the Universe. We find this by putting ds=0 into FRW and integrating it from t=0 till present:

$$ds=0 \;\rightarrow\; cdt = a(t)\frac{dr}{\sqrt{1-\kappa r^2}} \;\rightarrow\; \int_0^t \frac{cdt}{a(t)} = \int_0^{r_H}\frac{dr}{\sqrt{1-\kappa r^2}}$$

Thus

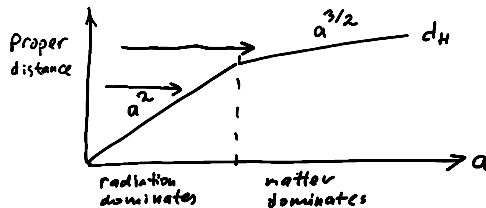$$d_H(t) = a(t)\int_0^t \frac{cdt}{a(t)}$$

For flat universe dominated by non-relativistic particles $a(t) \propto t^{2/3}$
In this case

$$d_H(t) = \frac{2c}{H_0} a^{3/2} = 3ct$$

For flat universe dominated by relativistic particles $a(t) \propto t^{1/2}$
And $d_H \propto a^2 \quad (d_H = 2ct)$

Important notion: distance to the horizon grows faster than wavelengths of waves, which expand together with the Universe. Thus, free waves were outside of the horizon at early moment and cascade inside the horizon at later moments.

Angular diameter distance

Angular diameter distance d_A is defined as the ratio of an object's physical transverse size to its apparent angular size. We assume that the physical size y of the object is fixed (not expanding with the Universe).
Then

$$d_A = y/\theta$$

We use FRW metric to track two photons emitted at t_{em} from either side of the object. The angle between the photons is simply the transverse distance y divided by the radial distance to the object at the moment of emission:

$$\theta = \frac{y}{a_{em}\int_0^{r_{em}}\frac{dr}{\sqrt{1-\kappa r^2}}}$$

In order to fine the coordinate distance r_{em} corresponding to the moment of emission, we use FRW metric, put ds=0 (for light) and integrate it from the moment of emission till the present moment. If t_U is the age of the Universe, then

$$\int_{t_{em}}^{t_U} \frac{c\,dt}{a(t)} = \int_0^{r_{em}} \frac{dr}{\sqrt{1-Kr^2}}$$

Thus,

$$d_A = a_{em} \int_{t_{em}}^{t_U} \frac{c\,dt}{a(t)}$$

Important case: $a \propto t^{2/3}$

$$t_U = \frac{2}{3} \frac{1}{H_0}$$

$$d_A = \frac{2c}{H_0} a_{em} \left(1 - \sqrt{a_{em}}\right)$$

Luminosity distance

Luminosity distance is defined as $\quad d_L = \sqrt{\dfrac{L_{Bol}}{4\pi f_{bol}}}$

Here L_{Bol} **is bolometric luminosity and** f_{bol} **is the observed bolometric flux.**

The radiation from the object traveled the distance

$$d = a(t) \int_{t_{em}}^{t_U} \frac{c\,dt}{a(t)}$$

Thus, it is spread of area $4\pi d^2$

Two other factors: energy of each photon decreases and the rate with which receive the photon also goes down: $\quad \hbar \nu_{now} = \hbar \nu_{em} a_{em}, \quad \Delta t_{now}/\Delta t_{em} = a_{now}/a_{em}$

This gives the observed flux:

$$f_{obs} = \frac{L_{Bol,obs} \cdot a_{em}}{4\pi d^2} = \frac{L_{Bol,em}}{4\pi d^2} a_{em}^2 \Rightarrow d = \sqrt{\frac{L_{Bol,em}}{4\pi f_{obs}}}\, a_{em}$$

$$\Rightarrow \boxed{d_L = d/a_{em}} \quad \text{and} \quad \boxed{d_L = d_A/a_{em}^2}$$