1. THE VIRIAL THEOREM: Proof

Set up: We have an isolated system of N objects. Masses, coordinates and velocities are given: $m_i, \vec{r_i}, \vec{V_i}$.

Now we are looking for a relation between the total kinetic energy and the total potential energy. The energies are:

$$\mathcal{K} = \frac{1}{2}\sum_i m_i({\dot{x}_i}^2 + {\dot{y}_i}^2 + {\dot{z}_i}^2), \quad \mathcal{W} = -\frac{1}{2}G\sum_{i,j}\frac{m_i m_j}{|\vec{r_i} - \vec{r_j}|} \tag{1}$$

We start with defining the moment of inertia:

$$\mathcal{I} = \sum_{i=1}^{N} m_i[{x_i}^2 + {y_i}^2 + {z_i}^2]. \tag{2}$$

We will differentiate $\mathcal{I}$ twice with respect to time t and then use equations of motion. At some moment we will also need to use the Newton's law of gravity.

$$\frac{d\mathcal{I}}{dt} = 2\sum m_i[x_i\dot{x}_i + y_i\dot{y}_i + z_i\dot{z}_i]. \tag{3}$$

Now the second derivative:

$$\frac{d^2\mathcal{I}}{dt^2} = 2\sum m_i[{\dot{x}_i}^2 + {\dot{y}_i}^2 + {\dot{z}_i}^2 + x_i\ddot{x}_i + y_i\ddot{y}_i + z_i\ddot{z}_i]. \tag{4}$$

Time to use the equations of motion:

$$\ddot{\vec{r_i}} = -G\sum_j \frac{m_j(\vec{r_i} - \vec{r_j})}{|\vec{r_i} - \vec{r_j}|^3}. \tag{5}$$

Take the last three terms in eq. 4 and substitute accelerations with their expressions in equations of motions:

$$\sum m_i(x_i\ddot{x}_i + y_i\ddot{y}_i + z_i\ddot{z}_i) = -G\sum_{i,j}\frac{m_i m_j}{|\vec{r_i} - \vec{r_j}|^3}\left[x_i(x_i - x_j) + y_i(y_i - y_j) + z_i(z_i - z_j)\right] \tag{6}$$

$$= -G\sum_{i,j}\frac{m_i m_j}{|\vec{r_{ij}}|^3}\left[{x_i}^2 + {y_i}^2 + {z_i}^2 - x_i x_j - y_i y_j - z_i z_j\right] \tag{7}$$

Note that in this double sum the index i is independent of index j. As the matter of fact, we can use any letter, the result is the same. In equation (7) we can swap the indexes: $i \Leftrightarrow j$. Thus,

$$\begin{aligned} S &\equiv \sum m_i(x_i\ddot{x}_i + y_i\ddot{y}_i + z_i\ddot{z}_i) = \\ &= -\frac{G}{2}\sum_{i,j}\frac{m_i m_j}{|\vec{r_{ij}}|^3}\left[{x_i}^2 + {y_i}^2 + {z_i}^2 - x_i x_j - y_i y_j - z_i z_j\right] - \\ &\quad -\frac{G}{2}\sum_{i,j}\frac{m_i m_j}{|\vec{r_{ij}}|^3}\left[{x_j}^2 + {y_j}^2 + {z_j}^2 - x_i x_j - y_i y_j - z_i z_j\right]. \end{aligned}$$

Collect terms with x_i: $x_i^2 - 2x_ix_j + x_j^2 = (x_i - x_j)^2$ and do the same for other components. We rewrite the sum S in a much more compact way:

$$S = -\frac{G}{2}\sum \frac{m_im_j}{r_{ij}^3} r_{ij}^2 = -\frac{G}{2}\sum \frac{m_im_j}{|\vec{r_i} - \vec{r_j}|} = \mathcal{W}. \quad (8)$$

Going back to eq. 4 we get:

$$\frac{d^2\mathcal{I}}{dt^2} = 2\mathcal{K} + \mathcal{W}. \quad (9)$$

This is the virial theorem. Typically it is applied with the assumption that the $d^2\mathcal{I}/dt^2 = 0$ implying a *stationary system*.

2. Limits of applicability

Two important details: (1) we did not do any averaging over time to come to the final expression. As long as the second derivative of the moment of inertia is zero, the virial theorem holds. In this form the virial theorem cannot be used for, say a binary star system on elongated orbit because the moment of inertia changes at each moment of time. (2) The kinetic and potential energies are *total* energies of all particles in the system. If we do not have information about some component of the system (e.g., dark matter) we cannot apply the theorem to estimate the mass of the system. We cannot even get the mass of the visible component because in our derivation we used gravitational accelerations, which depend on *all* matter, not only on its visible part. There is another complication: the system is supposed to be isolated. If it is not (as often is the case), the theorem cannot be used.

There are other ways to derive the virial theorem, which actually give somewhat different results. One typical situation is a system, which instantaneously not stationary, but when averaged over time gives $\langle d^2\mathcal{I}/dt^2\rangle = 0$. In this case the virial theorem reads:

$$2\langle\mathcal{K}\rangle + \langle\mathcal{W}\rangle = 0. \quad (10)$$

In other words, averaged over time kinetic and potential energies still obey the virial relation. Note that this is a different statement as compared with eq. (9) where the energies are *instantaneous* quantities. The time-average form of the virial theorem is not very useful in the case of galaxies or even clusters of galaxies because those systems evolve over time quite substantially. So, it is not clear over what period of time the averaging should be done. Nothing to say that observationally, time averaging of individual galaxies is unrealistic: should we wait a billion year to find the time-average of kinetic energy of our Milky Way galaxy? The instantaneous form is a bit easier to handle: we need to estimate the rate of change of the moment of inertia.

Virial theorems also are applicable to stars where the kinetic energy is the thermal energy of all gas particles. Quantum systems also obey the virial theorems.

3. Masses of objects and the virial theorem

We can use the virial theorem to estimate the mass of a stationary isolated system. We introduce the gravitational radius r_g of a system:

$$\mathcal{W} = -\frac{GM^2}{r_g} \tag{11}$$

Note that here we do not assume anything about the system. We do not assume that is spherical or its density is smooth; no assumptions. This equation simply states that potential energy scales as M^2. Details of the mass distribution are hidden in r_g. For a system with equal-mass points we have:

$$\frac{1}{r_g} = \langle\frac{1}{r_{ij}}\rangle. \tag{12}$$

Another example: sphere with homogeneous density distribution $\rho(r)$ =const and radius R. In this case $\mathcal{W} = -\frac{3}{5}GM^2/R$ and $r_g = 5R/3$. Note that in this case the gravitational radius is larger than the radius of the sphere. Kinetic energy can be parameterized as

$$\mathcal{K} = -\frac{M\langle V^2\rangle}{2}, \tag{13}$$

where $\langle V^2\rangle$ is the velocity dispersion. Again, no assumptions here: this is always valid. Assuming that the line-of-sight velocity dispersion is 1/3 of the 3d velocity dispersion, we get the following estimate of the mass:

$$M = \frac{3\langle V_{los}^2\rangle r_g}{G}. \tag{14}$$

Here we specifically assumed that the observed rms velocities are equal to the rms velocities of all material in the system, which may not be correct. However, experience with different systems (either analytical models or numerical simulations) shows that this is not a bad approximation. The main problem is to estimate r_g. There are a number of problems with the estimate. First, r_g typically is a radius, which is dominated by large-distance pairs. Each pair has a relatively small contribution, but there are many of them and they dominate the final result. Because observationally it is more difficult to measure objects in outer radii of a system, we are susceptible to numerous observational complications (e.g, projection or non-equilibrium effects). Second, spatial distribution of observed objects may be very different as compared with, say dark matter.

It is remarkable that the virial theorem still gives sensible estimates of masses. It is generally believed that it gives estimates within a factor of 2-3 of the real values. However, it is difficult to make those estimate more accurate.