1. Elements of Thermodynamics

Motivation: Equations of hydrodynamics are not "closed": they have a term with gas pressure, which cannot be found unless we know how it is related with other variables such as gas density and internal energy. However, equations of hydrodynamics are significantly simpler than the Jeans equations of stellar dynamics, where the pressure term is a tensor $\rho\sigma^2_{ij}$. In hydrodynamics pressure is a *scalar:* $\rho\sigma^2_{ij} = P\delta_{ij}$.

Later we will find that the presence of pressure term in the Euler equation brings in an additional equation = an equation of energy evolution.

Definitions:

- Specific volume
$$V \equiv \frac{1}{\rho}, \tag{1}$$
where ρ is the gas density.

- Pressure
$$P = \frac{\Delta F}{\Delta S}, \quad F = \frac{\Delta(mv)}{\Delta t}, \quad P = \frac{\Delta(mv)}{\Delta t \Delta S}. \tag{2}$$

- Temperature T
$$\frac{mv^2_{\rm thermal}}{2} = \frac{3}{2}kT, \quad k = 1.38 \cdot 10^{-16}{\rm erg/degree}. \tag{3}$$

- Relation between energy and temperature: $1eV = 1.6 \cdot 10^{-12}{\rm erg} = 11600{\rm degrees}$.

- Energy per unit mass ϵ. Energy per unit volume $u = \rho\epsilon = (3/2)nkT$ (for mono-atomic gas).

- Statistical weight Γ = number of microscopic states, which give the same macroscopic state. Entropy
$$S = k \ln \Gamma. \tag{4}$$
Entropy for collisionless particles can be defined as
$$S = -\int f \ln f d^3v. \tag{5}$$
H-theorem states that $dS/dt > 0$. Entropy is preserved along a trajectory.

The first law of thermodynamics:

$$\frac{d\epsilon}{dt} + P\frac{dV}{dt} = \frac{dQ}{dt}, \tag{6}$$

where dQ/dt is the energy per unit time per unit mass from external sources.

The second law of thermodynamics:

$$Tds = d\epsilon + pdV. \tag{7}$$

For an adiabatic process $dS/dt = 0$

Equation of state: $\epsilon = \epsilon(P, \rho)$

- Ideal gas:

  $$P = nkT, \tag{8}$$

  where n is the number-density of particles.

- Black-body radiation:

  $$P = \frac{\rho c^2}{3} = \frac{u}{3}, \quad u = \sigma T^4, \sigma = 7.57 \cdot 10^{-15}. \tag{9}$$

  Entropy $S \propto T^3$.

- Degenerate electron gas (for white dwarfs):

  $$P = 10^{13}(\rho/\mu_e)^{5/3}, \quad \mathrm{non-relativistic}, \tag{10}$$
  $$P = 1.2 \cdot 10^{15}(\rho/\mu_e)^{4/3}, \quad \mathrm{relativistic}, \tag{11}$$

  where μ_e is the molecular weight per electron (1-2)

More on ideal gas:

$$P = nkT. \tag{12}$$

Gas density ρ and molecular weights:

$$\begin{aligned} \rho &= \mu_m m_H n \qquad (13)\\ &= \mu_H m_H n_H, \qquad (14) \end{aligned}$$

where $m_H = 1.67 \cdot 10^{-24}$g and μ_m = molecular weight per particle, μ_H = molecular weight per hydrogen atom. For fully ionized hydrogen-helium plasma:

$$\begin{aligned} \mu_m &= \frac{n_H + 4n_{\rm He}}{2n_H + 3n_{\rm He}} = \frac{number - of - baryons}{number - of - particles} \qquad (15)\\ \mu_H &= \frac{n_H + 4n_{\rm He}}{n_H} \qquad (16) \end{aligned}$$

Helium abundance by mass Y is defined as

$$Y = \frac{4n_{\rm He}}{n_H + 4n_{\rm He}}. \tag{17}$$

For $Y = 0.25$ we have $n_{rmHe}/n_H = 1/12$, and $\mu_m = 16/27 \approx 0.6$, $\mu_H = 4/3$. Now we can write the equation of state for ideal gas as:

$$P = \frac{\rho k T}{\mu_m m_H}. \tag{18}$$

For mono-atomic gas the thermal energy ϵ is:

$$\epsilon = \frac{3}{2}\frac{nkT}{\rho} = \frac{3}{2}\frac{kT}{\mu_m m_H}. \tag{19}$$

Heat capacity: amount of heat, needed to raise temperature T by 1 degree:

$$\begin{aligned} c_V &= \frac{dQ}{dT}|_V = \frac{d\epsilon}{dT} = \frac{3}{2}\frac{k}{\mu_m m_H}, \qquad (20)\\ c_P &= \frac{dQ}{dT}|_P = \frac{\epsilon}{dT}|_P + p\frac{dV}{dT}|_P = \frac{5}{2}\frac{k}{\mu_m m_H}, \qquad (21)\\ c_P &= c_V + \frac{k}{\mu_m m_H} \qquad (22) \end{aligned}$$

The ratio of specific heats:

$$\gamma = \frac{c_P}{c_V} = 5/3. \tag{23}$$

For gas with an arbitrary γ:

$$\epsilon = \frac{1}{(\gamma - 1)}\frac{kT}{\mu_m m_H}. \tag{24}$$

Adiabatic process: no exchange with the outside world (no radiation, not energy release or conduction):

$$P = A\rho^{\gamma}, \tag{25}$$

where A is a constant. In this case the entropy is constant: $dS = 0$. Entropy of ideal gas can be introduced as

$$S = c_V \ln\left(P/\rho^{\gamma}\right). \tag{26}$$

Then,

$$P = e^{S/c_V}\rho^{\gamma}, \quad \epsilon = \frac{1}{(\gamma-1)}\frac{kT}{\mu_m m_H}. \tag{27}$$

For this process we can write the first law of thermodynamics in the following form:

$$\frac{d\epsilon}{dt} = -P\frac{d(1/\rho)}{dt} = \frac{P}{\rho^2}\frac{r\rho}{dt}. \tag{28}$$

When gas emits energy the last equation should be modified. It is more convenient to use energy per unit volume:

$$\frac{du}{dt} = \rho\frac{d\epsilon}{dt}, \tag{29}$$

$$\rho\frac{d\epsilon}{dt} = \frac{P}{\rho}\frac{d\rho}{dt} - n_e n_H \Lambda(T). \tag{30}$$

Adiabatic atmosphere: Assuming that the Earth atmosphere is adiabatic, find the density and temperature profiles.

Hydrostatic equilibrium:

$$\frac{1}{\rho}\frac{dP}{dz} = -g. \tag{31}$$

Equation of state:

$$P = A\rho^{\gamma}, \tag{32}$$

Parameter A (related to the entropy) is given by conditions at the surface:

$$P_0 = A\rho_0^{\gamma}, \quad P_0 = \rho_0 \frac{kT_0}{\mu m_H}. \tag{33}$$

For Earth's atmosphere $\mu = 29$, $\gamma = 1.4$. This gives:

$$A = \frac{kT_0}{\mu m_H}\rho_0^{1-\gamma}. \tag{34}$$

Rewrite the equation of hydrostatic equilibrium:

$$\frac{1}{\rho}\frac{dP}{dz} = \frac{A}{\rho}\gamma\rho^{\gamma-1}\frac{d\rho}{dz} = A\gamma\rho^{\gamma-2}\frac{d\rho}{dz} \tag{35}$$

Solution is:

$$\rho^{\gamma-1} = \rho_0^{\gamma-1} - \frac{gz}{A}\frac{\gamma-1}{\gamma}. \tag{36}$$

Rewrite this by dividing both parts of equation by $\rho_0^{\gamma-1}$ and substituting A:

$$\rho(z) = \rho_0\left[1 - \frac{z}{z_H}\right]^{1/(\gamma-1)}, \tag{37}$$

where the scale height is

$$z_H = \frac{\gamma}{\gamma-1}\frac{kT_0}{\mu m_H}\frac{1}{g}. \tag{38}$$

Now find temperature using the equation of state:

$$P = A\rho^{\gamma} = \rho\frac{kT}{\mu m_H}. \tag{39}$$

This gives:

$$T = T_0\left[\frac{\rho}{\rho_0}\right]^{(\gamma-1)}, \tag{40}$$

Substitute $\rho(z)$:

$$T = T_0\left[1 - \frac{z}{z_H}\right] \tag{41}$$

Taking $T_0 = 300K$, we get $z_H = 30$ km. This solution is valid only for $z < z_H$. It is not physical for high elevations!