Propagation of small-amplitude waves

Propagation of waves in gases: Sound waves and Jeans instability

- homogeneous gas: $\rho_0 = \text{const}$
  $V_0 = 0$
  $p_0 = \text{const}$
- isentropic gas: entropy is conserved
- propagation of small perturbations in the form $\Delta\rho$, Δp, Δv
- include gravity

Small variations in density $\Delta\rho$ result in variations in pressure Δp:

$$\Delta p = \left(\frac{\partial p}{\partial \rho}\right)_s \Delta\rho \equiv c^2 \Delta\rho$$

As we will see later c is the velocity of sound

Equations: hydro + Poisson:

$$\begin{cases} \frac{\partial \rho}{\partial t} + \nabla(\rho \vec{v}) = 0 \\ \frac{\partial \vec{v}}{\partial t} + (\vec{v}\nabla)\vec{v} = -\frac{\nabla p}{\rho} - \nabla\varphi = \frac{-c^2}{\rho}\nabla\rho - \nabla\varphi \\ \nabla^2\varphi = 4\pi G\rho \end{cases}$$

In the absence of fluctuations ($\Delta\rho = \Delta v = 0$) those equations are fulfilled:

$$v = v_0 = 0, \quad \rho = \rho_0 = \text{const}, \quad p = p_0 = \text{const}; \quad \nabla^2\varphi_0 = 4\pi G\rho_0$$

Now introduce small fluctuations:

$$\rho = \rho_0 + \Delta\rho, \quad \Delta\rho \ll \rho_0$$
$$v = \Delta v$$
$$P = P_0 + \Delta\rho \cdot c^2$$

Keeping only linear terms in our equations, we obtain:

$$
\begin{cases}
(1) & \dfrac{\partial \Delta\rho}{\partial t} + \rho_0 \nabla(\Delta v) = 0 \\
(2) & \dfrac{\partial \Delta v}{\partial t} = -\dfrac{c^2}{\rho_0}\nabla(\Delta\rho) - \nabla(\Delta\varphi) \\
(3) & \nabla^2(\Delta\varphi) = 4\pi G \Delta\rho
\end{cases}
$$

Take div of equation (2): $\dfrac{\partial}{\partial t}\nabla(\Delta v) = -\dfrac{c^2}{\rho_0}\nabla^2(\Delta\rho) - \nabla^2(\Delta\varphi)$

Now use (3):

$$(4) \qquad \frac{\partial}{\partial t}\nabla(\Delta v) = -\frac{c^2}{\rho_0}\nabla^2(\Delta\rho) - 4\pi G \Delta\rho$$

Take time derivative of (1) and use (4):

$$(5) \qquad \boxed{\frac{\partial^2 \Delta\rho}{\partial t^2} - c^2\nabla^2(\Delta\rho) - 4\pi G\rho_0 \Delta\rho = 0}$$

This is an equation for $\Delta\rho$. Try if the following gives a solution:

$$(6) \qquad \boxed{\Delta\rho = Const\cdot \exp(i(\vec{k}\cdot\vec{x} - \omega t))}$$

$$\Rightarrow \frac{\partial^2 \Delta\rho}{\partial t^2} = Const\cdot(i\omega)^2 = -Const\,\omega^2 \exp(i(\vec{k}\vec{x} - \omega t))$$

$$\nabla^2(\Delta\rho) = \frac{\partial^2}{\partial x^2}\Delta\rho + \frac{\partial^2}{\partial y^2}\Delta\rho + \frac{\partial^2 \Delta\rho}{\partial z^2}$$

$$\Rightarrow \frac{\partial^2}{\partial x^2}\Delta\rho = Const\,(ik_x)^2 = -Const\,k_x^2 \Rightarrow \nabla^2(\Delta\rho) = -Const\left[k_x^2 + k_y^2 + k_z^2\right] =$$

$$= -k^2\,Const\cdot\exp(i(\vec{k}\vec{x} - \omega t))$$

Substituting these expressions into (5):

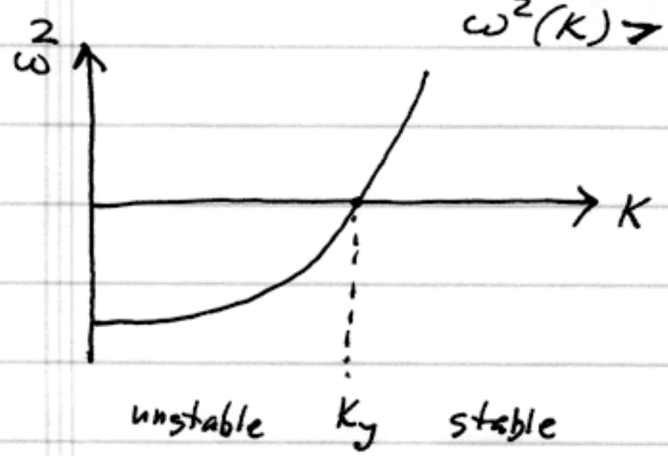
$$-\omega^2 + k^2c^2 - 4\pi G\rho_0 = 0$$

OR

$$\boxed{\omega^2 = -4\pi G\rho_0 + k^2c^2}$$

Any fluctuation can be decomposed into a sum of Fourier harmonics eq (6). If one of the harmonics is growing, than the system is unstable. Thus, the condition of stability is

$$\omega^2(k) > 0$$



Condition of stability:

$$k^2 c^2 > 4\pi G \rho_0$$

Critical regime $\omega^2 = 0$ defines the border of stability:

$$k_J^2 = \frac{4\pi G \rho_0}{c^2}$$

$$\lambda_J = \frac{2\pi}{k_J} = c\sqrt{\frac{\pi}{G\rho_0}} \; ; \; M_J = \frac{4\pi}{3}\rho_0 \left(\frac{\lambda_J}{2}\right)^3 =$$

$$= \frac{\pi^{5/2}}{6 G^{3/2}} \frac{c^3}{\rho_0^{1/2}} \propto \frac{T^{3/2}}{\rho^{1/2}}$$

For isoentropic ideal gas:

$$P = e^{S/c_v} \cdot \rho^{\gamma} \Rightarrow c^2 = \left.\frac{dP}{d\rho}\right|_S = \gamma \frac{P}{\rho}$$

$$P = \rho \frac{kT}{\mu m_H} \rightarrow \boxed{c^2 = \gamma \frac{P}{\rho} = \gamma \frac{kT}{\mu m_H}}$$

Why c is the sound velocity?

Let's simplify the situation and consider 1 dimensional plane flow. Neglect gravity.
Then equation (5) is simply

wave equation

$$\frac{\partial^2 \Delta\rho}{\partial t^2} = c^2 \frac{\partial^2 \Delta\rho}{\partial x^2} \Leftarrow \text{1D wave equation}$$

This wave equation has two families of solutions

$$\Delta\rho = \Delta\rho(x - ct) \quad \text{and} \quad \Delta\rho = \Delta\rho(x + ct)$$

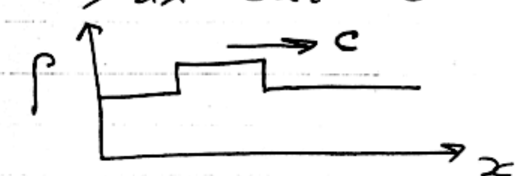
[introduce $y = x \pm ct$ as a new variable, then $\Delta\rho = \Delta\rho(y)$

$$\frac{\partial^2}{\partial t^2}\Delta\rho = c^2 \Delta\rho''; \quad c^2\frac{\partial^2 \Delta\rho}{\partial x^2} = c^2 \Delta\rho''$$]

The first expression gives a wave of a constant shape, which propagates in the positive x direction.
The second – wave goes in the negative x direction.
The velocity, with which the disturbance moves is defined by the condition:

$$x - ct = \text{const} \Rightarrow dx - c\,dt = 0$$

$$\boxed{c = \frac{dx}{dt}}$$



$c^2 = \frac{dP}{d\rho}$ is the square of the velocity of the wave.

Gas moves with different velocity. Find velocity of the gas:

Note the sign (±)

$$v = v(x \mp ct) \quad \text{[two solutions]}$$

$$\frac{\partial \Delta\rho}{\partial t} = -\rho_0 \frac{\partial v}{\partial x} \quad \text{(continuity equation)}$$

but $\frac{\partial v}{\partial x} = \mp \frac{1}{c}\frac{\partial v}{\partial t} \Rightarrow \boxed{\frac{\Delta\rho}{\rho_0} = \pm \frac{v}{c}}$

The upper sign is for wave propagating to the right and the lower sign is for wave moving to the left.

In both cases particles of the gas are moving in the same direction as wave if gas is compressed ($\Delta\rho > 0$) and are moving in oposite direction where gas is expanded ($\Delta\rho < 0$)

In general, we have a superposition of two waves

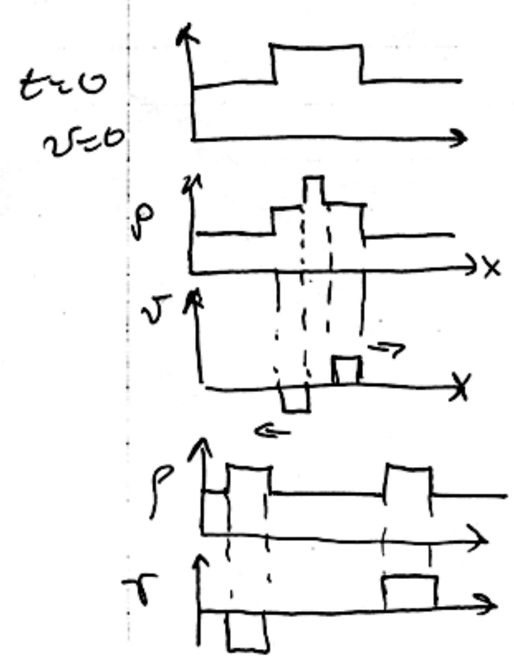
$$\Delta\rho = \rho_0 f_1(x-ct) + \rho_0 f_2(x+ct)$$

$$v = c f_1(x-ct) - c f_2(x+ct)$$

where f_1 and f_2 are functions determined by initial density and velocity:

$$f_1 = \frac{1}{2}\left[\frac{\Delta\rho(x,0)}{\rho_0} + \frac{v(x,0)}{c}\right]$$

$$f_2 = \frac{1}{2}\left[\frac{\Delta\rho}{\rho_0}(x,0) - \frac{v(x,0)}{c}\right]$$



Characteristics: 1-dimensional isoentropic flow

If unperturbed gas is at rest, then arbitrary small disturbances will travel in both directions (positive and negative x). For a perturbation traveling to the right (positive x):

$$\frac{\Delta\rho_1}{\rho_0} = \frac{v_1}{c} = f_1(x-ct)$$

Here ρ_0 is unperturbed density and c is the velocity of sound ($c^2 = \frac{dP}{d\rho}$). For a perturbation moving to the left:

$$\frac{\Delta\rho_2}{\rho_0} = -\frac{v_2}{c} = f_2(x+ct).$$

An arbitrary perturbation can be decomposed in two waves: one traveling to the left and another to the right

$$v = v_1 + v_2 \; ; \quad \Delta\rho = \Delta\rho_1 + \Delta\rho_2$$

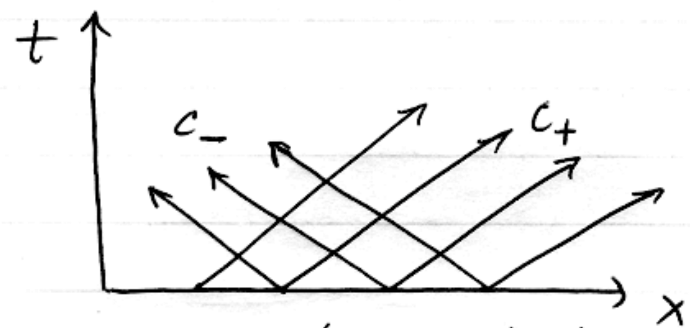
What happens if the background moves with some velocity u? We can chose the reference frame, which moves with the same velocity u. In this frame the background is not moving and we have the same situation as before: two waves moving to the left and to the right:

In the frame, which does not move the waves are moving with velocities $u \pm c$:

$$\frac{dx}{dt} = u + c \quad \text{and} \quad \frac{dx}{dt} = u - c \qquad (*)$$

Equations (*) define two families of curves in the plane (x,t):

C_+: $\frac{dx}{dt} = u + c$ and C_-: $\frac{dx}{dt} = u - c$

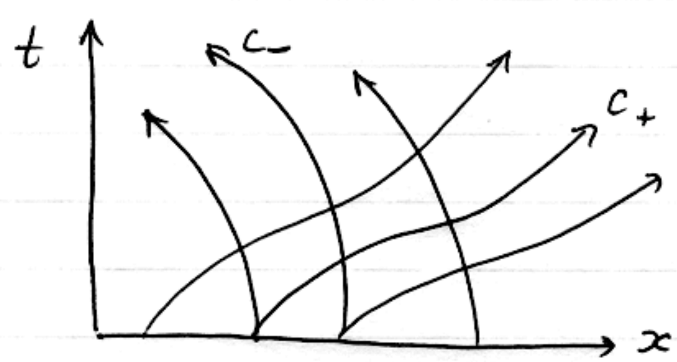


Two curves: C_+ and C_- pass through every point (t, x)

Now we make the situation more complicated: parameters of the fluid (c, u) vary from point to point: $u = u(x,t)$, $c = c(x,t)$, $\rho_0 = \rho_0(x,t)$

Locally, we still have $\rho_0 \simeq$ const, $u \simeq$ const ...
Thus, locally we still have two waves: $\frac{dx}{dt} = u \pm c$

Because u and c can change from point to point, characteristics $C_+(x,t)$ and $C_-(x,t)$ are not straight lines but curves:



For $u=0$ and $c=$ const along each characteristic some combinations of physical parameters were preserved:

$f_1 = \frac{\Delta\rho_1}{\rho} = \frac{v_1}{c}$ (for C_+) and $f_2 = \frac{\Delta\rho_2}{\rho} = -\frac{v_2}{c}$ (for C_-)

In more general case there are two invariants, which stay constant along characteristics:

Riemann invariants

$$du + \frac{1}{\rho c}\,dp = 0 \quad \text{along } C_+$$

$$du - \frac{1}{\rho c}\,dp = 0 \quad \text{along } C_-$$

$$J_+ = u + \int \frac{c\,d\rho}{\rho}$$

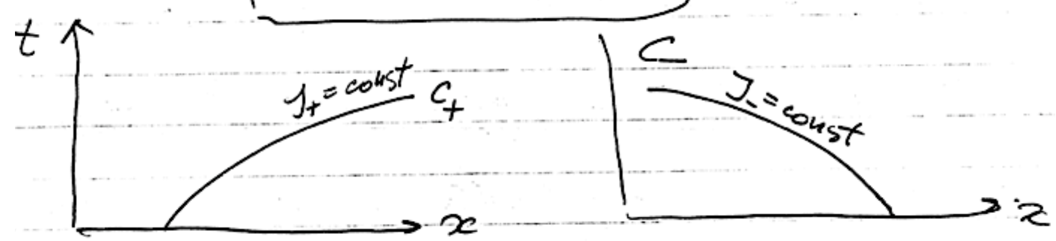
$$J_- = u - \int \frac{c\,d\rho}{\rho}$$

J_+ is constant along C_+: $\frac{dx}{dt} = u + c$

J_- is constant along C_-: $\frac{dx}{dt} = u - c$

for ideal gas: $p = A\cdot\rho^{\gamma}$; $c^2 = \gamma\cdot A\rho^{\gamma-1}$
and

$$\boxed{J_{\pm} = u \pm \frac{2}{\gamma-1}\,c}$$

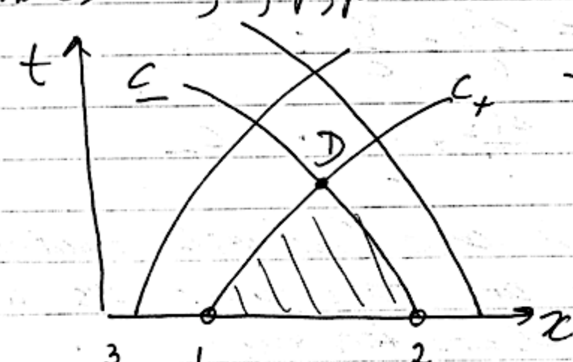


Note that the state of the gas is defined by two variables. We can choose u and ρ, say
or c and u
or J_+ and J_- :

$$u = \frac{J_+ + J_-}{2}; \quad c = \frac{\gamma-1}{4}\left(J_+ - J_-\right); \quad c^2 = \frac{dp}{d\rho}$$

Thus, if we know J_+ and J_-, we can get our "usual" variables u, c, p, ρ

Domain of dependance

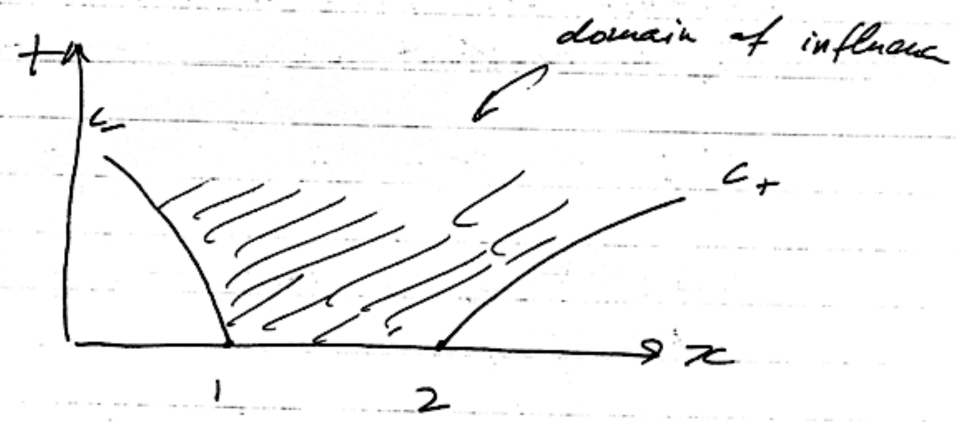


the gas in D is defined by $J_+(x_1, 0)$ and $J_-(x_2, 0)$

but characteristics C_+ and C_- (going from 1 and 2) are themselves defined by conditions on $x_1 < x < x_2$

Important: any disturbance in 3 will NOT affect the state of gas in D

Region of influence



Characteristics $C_+(x_2, 0)$ and $C_-(x_1, 0)$ define region, which can be affected by some event, which happens inside $x_1 < x < x_2$ range. Points outside the region cannot be affected.

Condition for a wave to travel to the right only: $J_- = const$

Waves of finite amplitude

For waves of finite amplitude the situation is more complicated because different parts of the wave move with different velocity. Non linear nature of hydro equations also complicates the situation.

But we still have waves and characteristics. Let's assume that at initial moment

$$J_-(x,0) = const$$

Then the only motion will be the wave moving in positive x direction.

In this case characteristics are especially simple: they are just straight lines. Indeed, equations for characteristics $C_\pm$ can be written as:

$$\frac{dx}{dt} = u \pm c = F_\pm(J_+, J_-) \text{ ; where } F_\pm \text{ is a function of } J_\pm \text{ only}$$

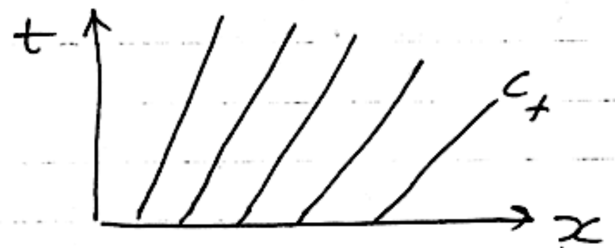
For ideal gas $F_+ = \frac{\gamma+1}{4} J_+ + \frac{3-\gamma}{4} J_-$, $F_- = \frac{3-\gamma}{4} J_+ + \frac{\gamma+1}{4} J_-$

[remember that $J_\pm = u \pm \frac{2}{\gamma-1} c$]

Because we have chosen $J_- = \text{const}$ and J_+ is constant along characteristics C_+

$$C_+: \quad x = F_+(J_+, J_-)t + \varphi(J_+)$$

where $\varphi(J_+)$ is a constant of integration



From $J_- = u - \int \frac{dp}{\rho c} = \text{const}$ follows that

$$c = c(u), \; p = p(u), \; \rho = \rho(u)$$

Then characteristic is simply

$$C_+: \quad x = [u + c(u)]t + \varphi(u)$$

This means that given values of u and $c(u)$ are carried through the gas with constant velocity

$$u + c(u)$$

Thus, the solution of hydro equations is a $\boxed{\text{wave}}$ traveling to the right:

$$u = f(x - [u + c(u)]t), \quad c = g(x - [u + c(u)]t)$$

As before, the shape of f and g are defined by initial conditions.