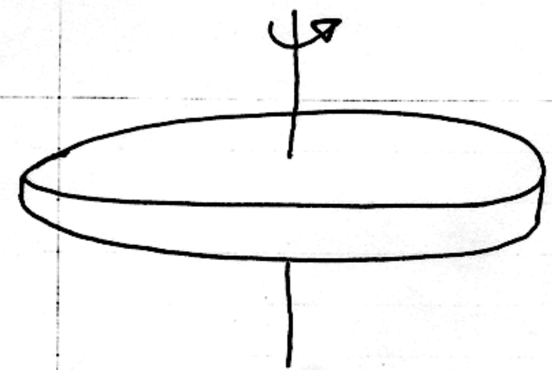
Instabilities:
Toomre and bending

Stability of a thin disk with differential rotation



$\varkappa$ = epicycle frequency

Σ = surface density of the disk

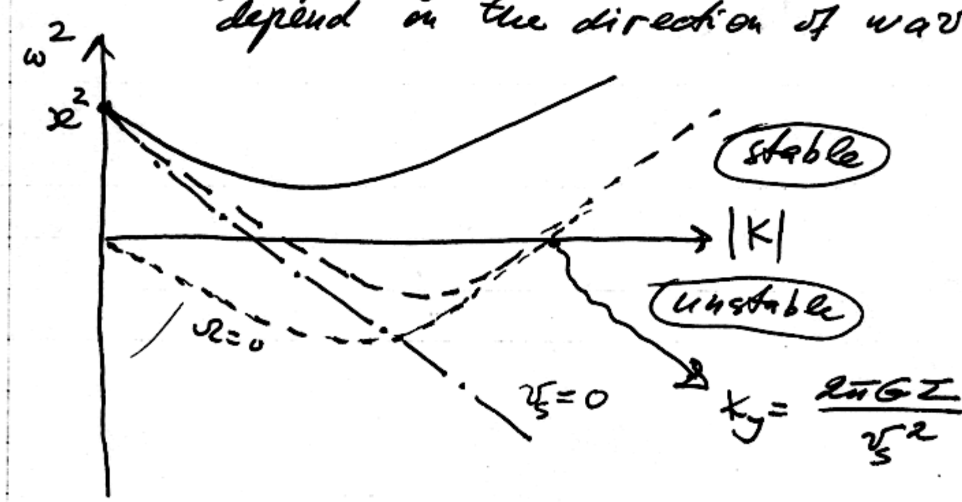
v_s = sound velocity

σ_R = velocity dispersion of stars in radial direction

dispersion relation for gaseous disk:

$$\boxed{\omega^2 = \varkappa^2 - 2\pi G\Sigma |k| + k^2 v_s^2}$$

Note that there are waves which travel along and against the disk rotation, but frequency of their oscillation does not depend on the direction of wave propagation



Condition for stability: $\omega^2 > 0$ at k defined by $\frac{d\omega^2(k)}{dk} = 0$

$$\frac{d\omega^2}{dk} = 0 \Rightarrow k = \frac{\pi G\Sigma}{v_s^2}; \quad \omega^2(k) = \varkappa^2 - \left(\frac{\pi G\Sigma}{v_s}\right)^2 \geq 0$$

$$\boxed{\varkappa > \frac{\pi G\Sigma}{v_s}}$$

Example: $\Omega = \frac{220\,\text{km/s}}{8.5\,\text{kpc}} = 26\,\text{km/s/kpc}$

$\varkappa \simeq 1.5\Omega \simeq 39\,\text{km/s/kpc}$

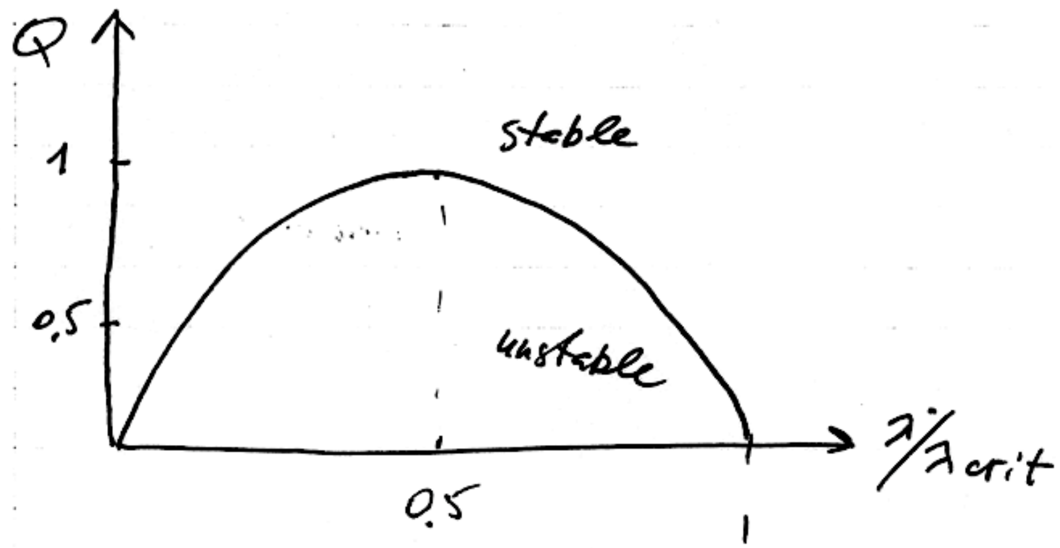
$v_s = 30\,\text{km/s}$, $\Sigma = 70\,M_\odot/\text{pc}^2$

$\pi G\Sigma / v_s = 31\,\text{km/s/kpc}$

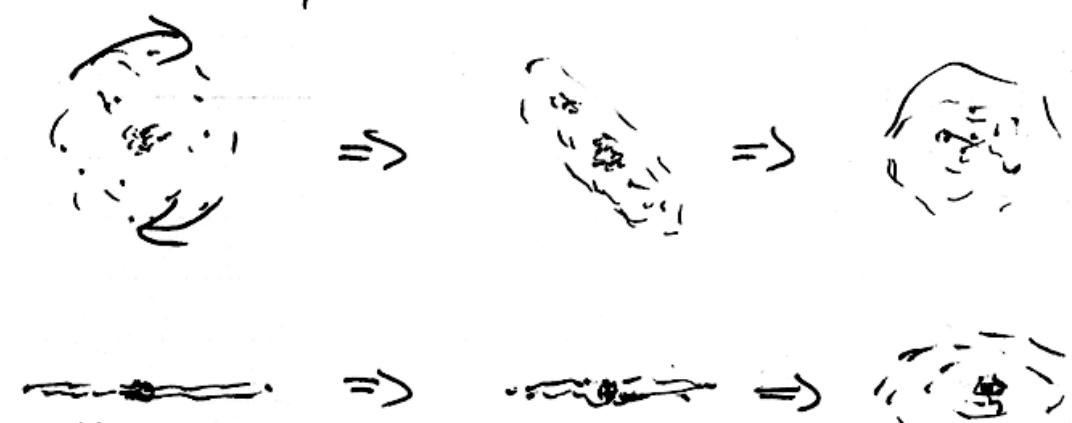
$$Q \equiv \frac{v_s \varkappa}{\pi G \Sigma} > 1 \text{ for stability (gas)}$$

$$Q \equiv \frac{\sigma_R \varkappa}{3.36\, G \Sigma} \quad \text{(stars)}$$

$$\lambda_{crit} \equiv \frac{4\pi^2 G \Sigma}{\varkappa^2}$$



Bar instability :



Ostriker - Peebles criterion :

$$\begin{matrix} \text{kinetic energy} \longrightarrow \\ \text{potential} \longrightarrow \end{matrix} \quad \frac{T_{Rotation}}{|W|} < 0.15 \text{ for stability}$$

modern simulations : buldge : disk : halo =

= 1 : 3 : 16

Formation of a bar

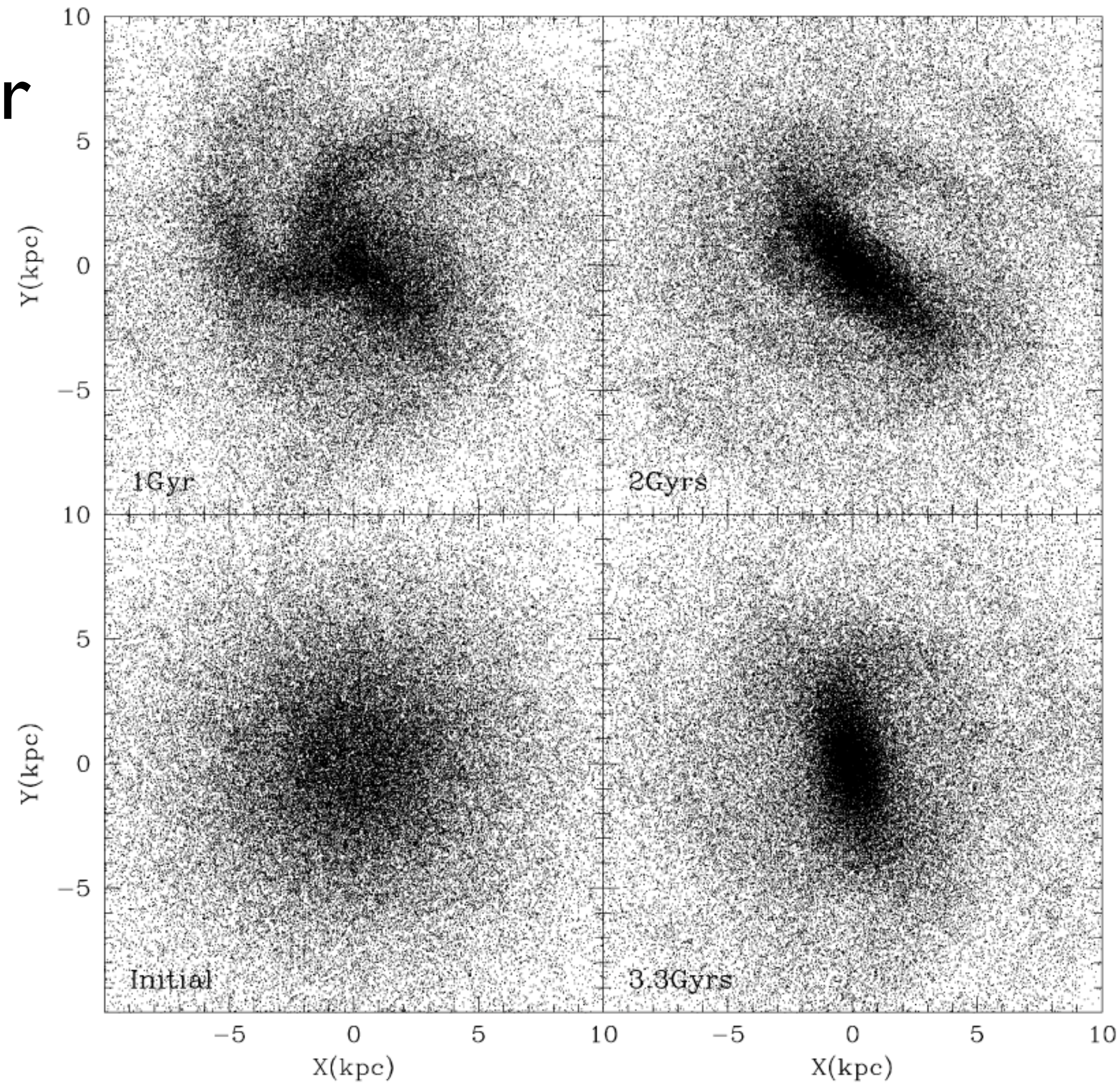
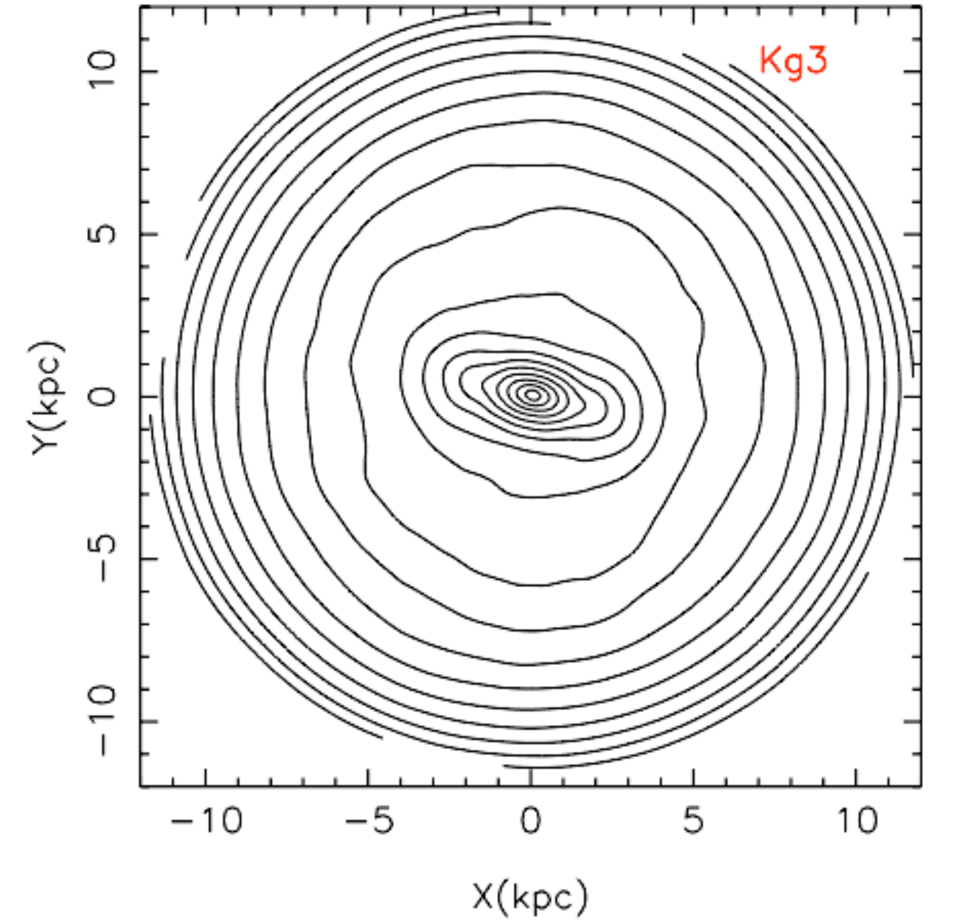


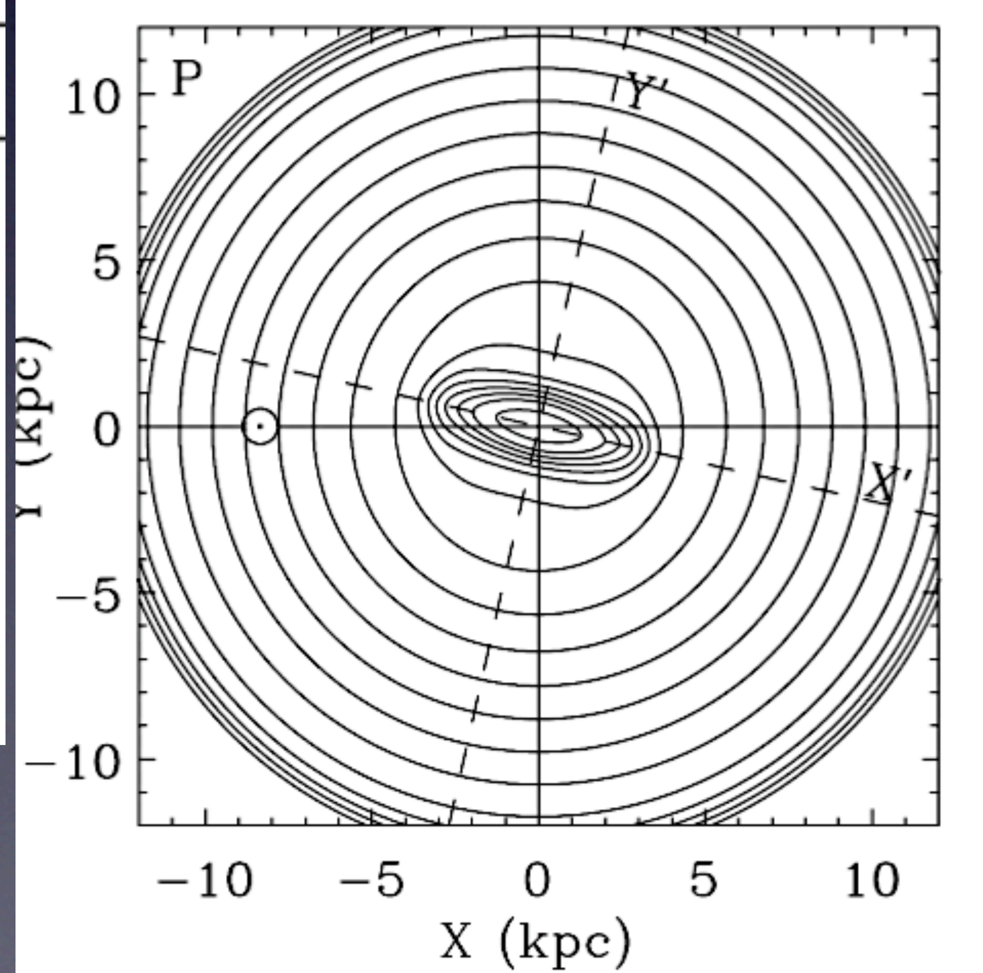
Figure 7. Distribution of the stellar component at different stages of evolution for model A_1. Starting from the bottom left-hand panel in the clockwise direction, the panels correspond to the initial moment, 1 Gyr, 2 Gyr and 3.3 Gyr. The disc rotates counter-clockwise. To avoid crowding we show only every fifth particle. At a distance of 5 kpc the disc made 20 orbital periods during 3.3 Gyr.

Matching Milky Way

Parameter	K_{g3}	Milky Way
Circular velocity (km/s)	220	210-230
Surface disk density at $R_\odot$ ($M_\odot/pc^2$)	44.6	48 ± 9
Vertical rms velocity of stars at $R_\odot$ (km/s)	14	15-20
Radial rms velocity of stars at $R_\odot$ (km/s)	38	35-40
Pattern Speed Ω_p (km/s/kpc)	50	53 ± 3
Bar length (kpc)	3.3	3.0-3.5
Total mass inside 60 kpc ($10^{11} M_\odot$)	5.5	4 ± 0.7
Total mass inside 100 kpc ($10^{11} M_\odot$)	7.3	7 ± 2.5

Klypin et al 2009





Bending - Bulking - firehose instability

Thin stellar systems in which some support against gravitational collapse comes from random motions are susceptible to a bending instability.
Following are examples where the bending instability plays important role:

- → Elliptical galaxies. The most flattened E have type E7. Anisotropic random velocities are responsible for flattening of most of elliptical galaxies. Why there are no E's which are more flattened than E7?
- → Bars in galaxies go through a stage called "bulking" resulting in formation of peanut-shape bulges.

Simple system illustrates how bending modes get unstable. Consider a thin disk with some velocity dispersion in the plane of the disk.

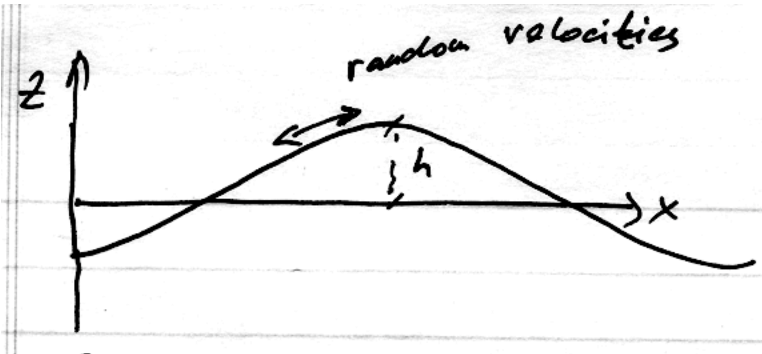
← velocity has only x component

Now we slightly perturb the disk by bending it in z direction. The perturbation has a form:

$$h(x,t) = H \exp(i(kx - \omega t))$$

↑ displacement in z-direction.

Stability of the perturbed system is defined by condition that $\omega^2 > 0$



$h = h(x,t)$ = vertical perturbation

The vertical acceleration of a particle is defined by two terms: the frequency of oscillation of the wave $\frac{\partial^2 h}{\partial t^2}$ and by the centrifugal force, which the particle "feels" when it moves along the wave: v^2/R, where R is the curvature of the wave:

$$\frac{v^2}{R} \simeq v^2 \frac{\partial^2 h}{\partial x^2}$$

The sum of the two terms must be equal (for stability case) to the force of gravity

$$\frac{\partial^2 h}{\partial t^2} + v^2 \frac{\partial^2 h}{\partial x^2} = g_z \qquad (*)$$

The "restoring" force of gravity is approximately equal to

$$g_z = +2\pi G \Sigma \cdot k \cdot h(x,t),$$

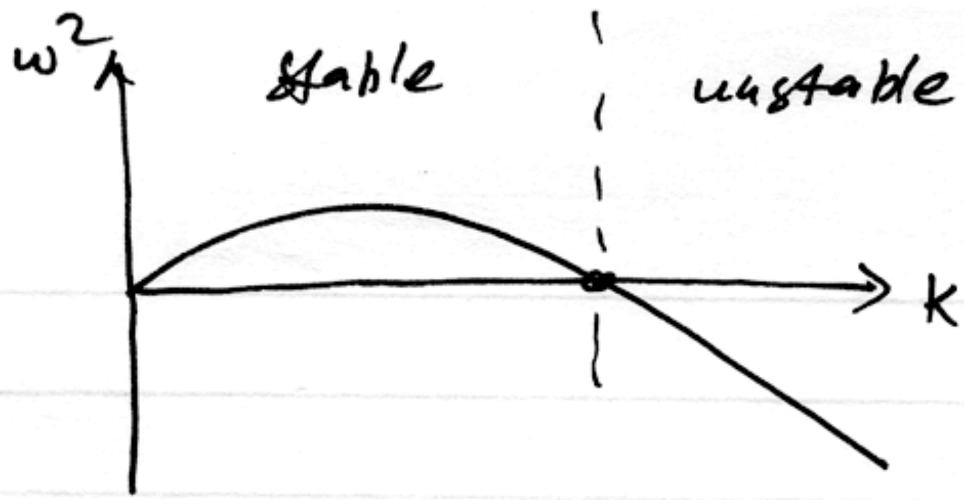
where $k = \frac{2\pi}{\lambda}$ = wavenumber of the perturbation

and Σ is the surface density of the disk

Equation $(*)$ gives us dispersion relation for the wave:

$$\boxed{\omega^2 = 2\pi G \Sigma k - v^2 k^2}$$

Note that the second term is the de-stabilizing the system while the first term (force of gravity) makes the system more stable



The hose instability operates on short scales
Corrections for finite disk thickness are quite complicated. One naively expects that vertical velocities stabelize the waves. If v_z is the rms velocity is z direction, then for stability

$$v_z > 0.3\, v$$

In reality the system becomes unstable when the velocity dispersions are larger. $\frac{v_z^2}{v_{radia}^2} \lesssim 0.3-0.5$