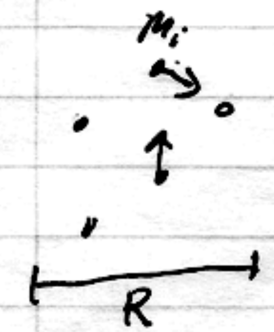
Two-body scattering

Two-body relaxation / scattering

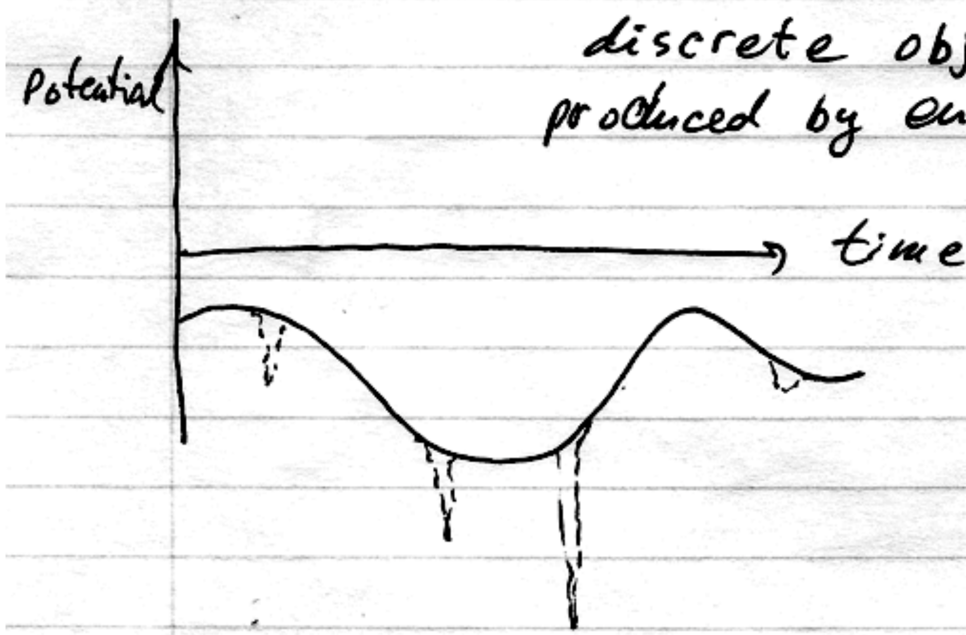
There are N objects with masses m_i moving with velocities $\vec{v}_i$. The objects are points – no sizes. The objects are moving in a system ("galaxy") of radius R and mass M



Potential, which an object "sees" as it moves through the system, has two components:

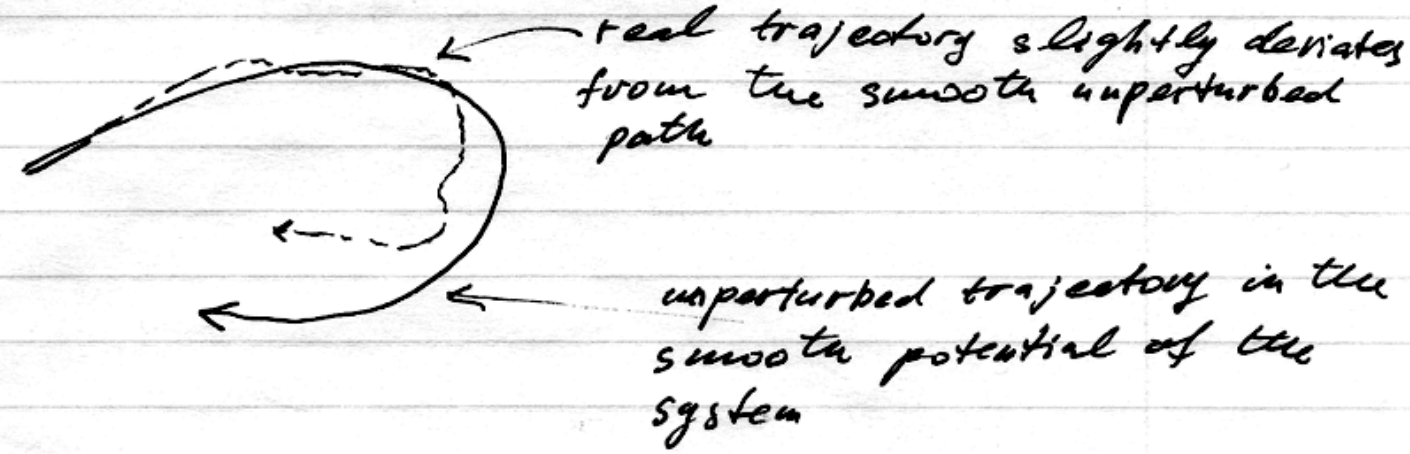
- a smooth potential due to the mass of all objects. This potential is a cumulative effect of masses in the system

- short and strong fluctuations due to the discrete objects. Those fluctuations are produced by encounters with individual objects: – "collisions"



There are a number of effects related and produced by the encounters:

⊖ Gradual accumulation of deviations of trajectories of particles. The time-scale of this accumulation is called the <u>two-body</u> <u>relaxation</u> <u>time</u>



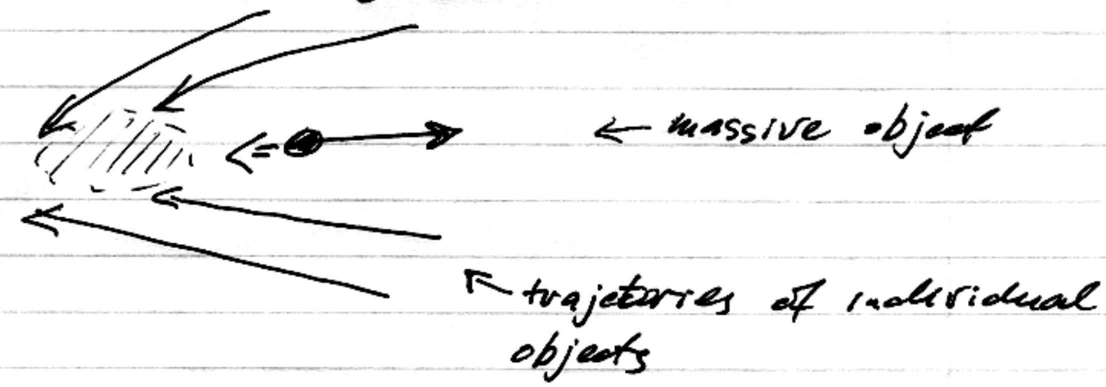
There are different ways to characterize the deviations. All of them treat the deviations in a <u>statistical</u> fashion.

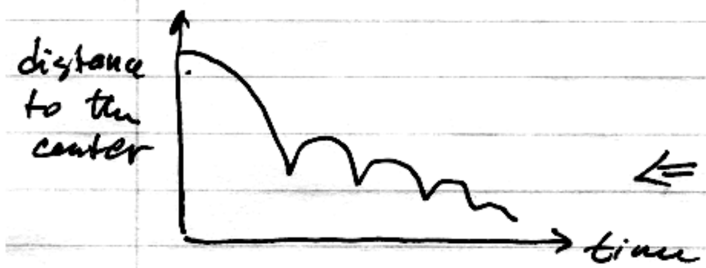
For example, we can define the relaxation time as the time when the <u>average</u> deviation of the velocity is equal to the velocity of unperturbed motion. Here "average" implies averaging over different trajectories. Another measure of deviations is the deviation of the energy of particles.

In order to find the time-scale for the effect, we estimate the local rate of the change of the measure of the deviations. For example, if we use velocities, then the time of the two-body relaxation is defined as

$$\frac{\Delta v}{v} = \frac{\Delta t}{t_{relax}}, \quad \text{where } \Delta v \text{ is the change in } v \text{ in } \Delta t \text{ interval}$$

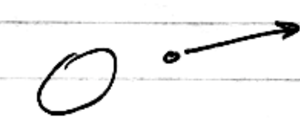
⊖ objects which are more massive than the typical mass of objects, experience a "drag" – their motion is slowed down. This effect is called <u>the dynamical friction</u>





⇐ massive object loses energy and angular momentum. It gradually sinks to the center of the system

⊖ If some objects slow down their motion, some will accelerate ("heating"). In extreme cases, this leads to "evaporation" – some objects escape from the system and became unbound



⊖ The process of slowing down of massive objects and accelerating of lighter objects leads to:

⊖ tendency of equipartition of energy

$$\left\langle \frac{m_1 v_1^2}{2} \right\rangle = \left\langle \frac{m_2 v_2^2}{2} \right\rangle$$

Note that this is just an instantenious tendency

⊖ mass segregation $\langle r \rangle \big|_{\text{massive objects}} < \langle r \rangle \big|_{\text{light objects}}$

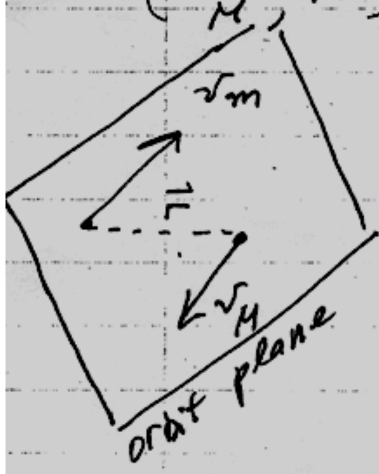
Two-body relaxation time

a star of mass M moves with velocity $\vec{v}_M$ through a "sea" of stars. The number-density of stars is n_0 and their velocity distribution is $f(v)$.
Later we will specify $f(v)$.
Due to gravitational encounters with the stars, the trajectory of our star M gradually changes. We need to estimate the rate of the change.

n:

1) Find results of an individual encounter
2) Sum up the results of many encounters

$(\vec{v}_M, M)$ encounters with $(\vec{v}_m, m)$



The equation for $\vec{r} = \vec{r}_m - \vec{r}_M$ is

$$\frac{mM}{m+M} \cdot \ddot{\vec{r}} = -\frac{GMm}{r^2}\vec{e}_r$$

if $\Delta\vec{v}_m$ and $\Delta\vec{v}_M$ are the changes in the velocities $\vec{v}_m$, $\vec{v}_M$ during the encounter, then

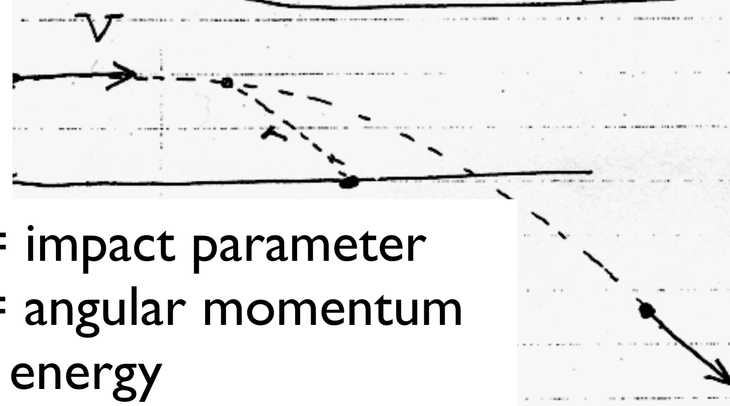
$$\Delta\vec{v}_m - \Delta\vec{v}_M = \Delta\vec{V}, \qquad \vec{V} = \dot{\vec{r}}$$

we also know (see "reduced mass" above) that

$$\Delta\vec{v}_M = -\frac{m}{m+M}\Delta\vec{V}$$

Thus, we need to find $\Delta\vec{V}$ after the collision.

Motion in the orbit plane:



b = impact parameter
L = angular momentum
E= energy

For Keplerian motion:

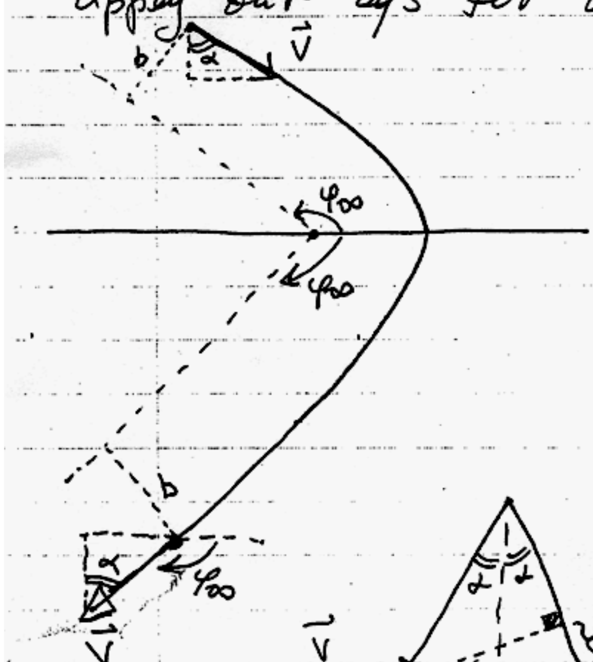
$$\frac{p}{r} = 1 + e\cos\varphi$$

$$e^2 = 1 + \frac{2EL^2}{(G(M+m))^2}$$

when the particle is at very large distance ($r = \infty$), we find **E** and **L**:

$$E = \frac{\mathbf{V}^2 mM}{2(m+M)} \qquad L = bV$$

apply our eqs for the keplerian motion:



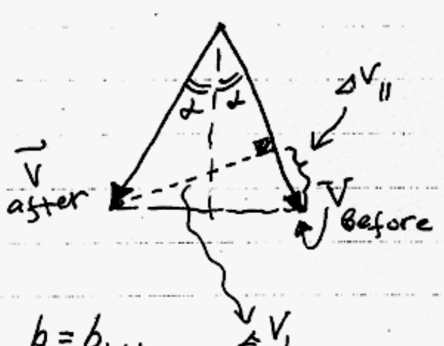
$$|\vec{V}|_{before} = |\vec{V}|_{after}$$

$$\alpha = \varphi_\infty - \frac{\pi}{2}$$

$$\frac{p}{r} = 1 + e\cos\varphi \Rightarrow \boxed{\cos\varphi_\infty = -\frac{1}{e}}$$

$$\Downarrow$$

$$\begin{cases} \sin\alpha = \dfrac{1}{e} \\ \cos\alpha = \sqrt{1-\sin^2\alpha} = \dfrac{\sqrt{e^2-1}}{e} \end{cases}$$



$$\begin{cases} \Delta V_\perp = V\sin 2\alpha = \dfrac{2bV^3}{G(m+M)}\dfrac{1}{e^2} = \\ = \dfrac{2bV^3}{G(m+M)}\dfrac{1}{1+\left(\dfrac{bV^2}{G(m+M)}\right)^2} \\ \Delta V_\parallel = V(1-\cos 2\alpha) = \dfrac{2V}{e^2} = \\ = \dfrac{2V}{1+\left(\dfrac{bV^2}{G(m+M)}\right)^2} \end{cases}$$

$\Delta V_\perp = V$ when $b = b_{min}$

$$\boxed{b_{min} = \frac{G(m+M)}{V^2}}$$

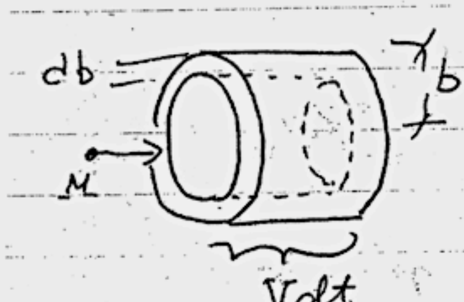
From those expressions, multiplying ΔV's by $\frac{m}{m+M}$, we obtain the chages of the velocity $\vec{v}_M$

$$|\Delta v_{M\perp}| = \frac{2mbV^3}{G(m+M)^2} \frac{1}{1+\left[\frac{bV^2}{G(m+M)}\right]^2} = 2\cdot\frac{m}{m+M}\cdot\frac{b}{b_{min}}\cdot V\frac{1}{1+\left(b/b_{min}\right)^2}$$

$$|\Delta v_{M\parallel}| = \frac{2mV}{M+m} \frac{1}{1+\left[\frac{bV^2}{G(m+M)}\right]^2} = 2\cdot\frac{m}{m+M}\cdot V \frac{1}{1+\left(b/b_{min}\right)^2}$$

We will use $\Delta v_{M\parallel}$ to estimate the dynamical friction. But now we are interested in the first equation, which gives us the time-scale for two-body relaxation

<u>How many collisions</u> we have <u>in dt interval</u>?



relative velocity

$f(\vec{v})d\vec{v}$ = number of stars in unit volume element with velocity $\vec{v}$ in the phase-space interval $d\vec{v}$

$$\left(\Sigma \Delta v_{M\perp}\right)^2 = \iiint f(\vec{v})d\vec{v}\cdot 2\pi b\,db\cdot\left(\Delta v_{M\perp}\right)^2 \times V dt$$

Note that here we integrate $\int(\Delta v_\perp^2)\ldots$, not $\Delta v_\perp$. The integral with the first power of $\Delta v_\perp$ is zero. We use the random walk approximation to estimate the rate of change in $\Delta v_\perp$

Let's take the integral over the impact parameter b.

Two types of collisions:

$b_{min} = \frac{G(m+M)}{V^2}$

- close: $b \leq b_{min} \rightarrow \Delta V_\perp \geq V$

Those collisions happen very rare.

For our solar neighborhood:

$m \simeq 1 M_\odot = 2 \cdot 10^{33}$

$V \simeq 20\ km/s = 2 \cdot 10^6$

$$b_{min} = \frac{6.67 \cdot 10^{-8} \cdot 2 \cdot 10^{33}}{(2 \cdot 10^6)^2} \frac{1}{3.08 \cdot 10^{18}} \simeq 10^{-5}\ pc\ !!$$

- long-distance: $b > b_{min} \rightarrow$ each collision has only very small impact, but there are many of them

$$I \equiv 2\pi \int_{b_{min}}^{\infty} db \cdot b \cdot (\Delta V_\perp^2) = 2\pi \int_{b_{min}}^{\infty} \left\{ \frac{2m}{m+M} V \right\}^2 \cdot b \cdot \left(\frac{b}{b_{min}}\right)^2 \frac{db}{\left[1 + \left(\frac{b}{b_{min}}\right)^2\right]^2} =$$

$x \equiv \frac{b}{b_{min}}$

$$= 2\pi \cdot \left\{ \frac{2m}{m+M} V \right\}^2 b_{min}^2 \int_1^{\infty} \frac{x^3 dx}{[1+x^2]^2} \approx 2\pi \left\{ \frac{2m}{m+M} V \right\}^2 b_{min}^2 \frac{1}{2} \ln\left(1 + \left(\frac{b_{max}}{b_{min}}\right)^2\right)$$

The integral $\int \frac{x^3 dx}{(1+x^2)^2}$ diverges at the upper limit of integration.

The divergence is very slow ($\sim$ logarithmic).

The integral should be truncated at 2-3 mean distances between stars. Reason: when we have many collisions at the same time our approximation (Keplerian motion) brakes.

$b_{max} \leftarrow$ upper limit for the impact parameter

Usually we take $b_{max} \simeq R \leftarrow$ radius of the system

$$\boxed{b_{max} \simeq R}$$

Final steps: integration over velocities

the integral I can be rewritten in the form:

$$I = 2\pi \left(\frac{2m}{m+M} V \right)^2 b_{min}^2 \, \frac{1}{2} \ln\left(1 + \frac{b_{max}}{b_{min}} \right) \approx$$

$$\simeq 8\pi \frac{G^2 m^2}{V^2} \ln \Lambda$$; where we used $b_{min} = \frac{G(m+M)}{V^2}$

and $\boxed{\Lambda = \frac{b_{max}}{b_{min}}}$

the term $\ln \Lambda$ is called Coulomb logarithm

Coulomb logarithm

	Solar neighborhood	Globular cluster	Cluster of galaxies
$\ln \Lambda$	18	10–12	6–9

$$\left\langle \Sigma \left(\Delta v_{M\perp} \right)^2 \right\rangle = 8\pi G^2 m^2 \int f(\vec{v}) \frac{d\vec{v}}{V} \cdot \ln \Lambda \cdot dt$$

here $V^2 = v_m^2 + v_M^2 - 2 v_m v_M \cos\theta$, where θ = angle between $\vec{v}_M$ and $\vec{v}_m$

In order to estimate the cumulative effect of collisions we must assume some form for $f(\vec{v})$. Observations in our solar neighborhood are consistent with a Maxwellian distribution:

$$\boxed{f(\vec{v}) = \frac{n}{\left[2\pi \sigma_v^2 \right]^{3/2}} \exp\left\{ -\frac{1}{2} \frac{v^2}{\sigma_v^2} \right\} v^2}$$

Itegration over v and θ gives

$$\boxed{\frac{\Sigma \left[\Delta v_\perp \right]^2}{v_\perp^2} = \frac{dt}{T_{relaxation}}}$$, where $T_{relaxation}$ is equal to:

Here: $\sigma_v \equiv v$

$$\boxed{T_{relaxation} \simeq \frac{v^3}{4\pi\, n\, m^2 G^2 \ln \Lambda}}$$

$$\Lambda = \frac{v^2}{G(m+M)} \cdot b_{max}$$

For a system, where particles collide with typical velocity

$$v^2 = \frac{G M_{tot}}{R} = \frac{G N \cdot m}{R},$$

N = number of objects.

$$\boxed{T_{relaxation} \simeq \frac{N}{8 \ln N} \cdot t_{dyn}} \qquad \underline{\underline{t_{dyn} = \frac{R}{v}}}$$

Object	Number N mass m	Radius R	rms Velocity v	Dynamical Time t_{dyn}	Relaxation time t_{rel}	age
Open clusters	~ 250 $1 M_\odot$	2pc	~ 1 km/s	$2 \cdot 10^6$ yrs	$5 \cdot 10^7$ yrs	$7 \cdot 10^8$ (Hyades) 10^7 yrs (χ Per) $5.5 \cdot 10^9$ yr (NGC 188)
Globular Clusters	$\sim 10^6$ $1 M_\odot$	10pc	~ 10 km/s	10^6 yrs	$2 \cdot 10^9$ yrs	~ 13 Gyrs
Solar Neighborhood	$0.1 \, pc^{-3}$ $1 M_\odot$	10kpc	20km/s (220 km/s)	$2 \cdot 10^8$ yrs	$2 \cdot 10^{13}$ yrs	~ 13 Gyrs
Galaxy Groups	20 $10^{11} - 10^{12} M_\odot$	1 Mpc	150 km/s	$5 \cdot 10^9$ yrs	$2.7 \cdot 10^{11}$ yrs $2.7 \cdot 10^9$ yrs	~ 10 Gyrs
Galaxy Clusters	10^3 $10^{11} - 10^{12} M_\odot$	2Mpc	~ 1000 km/s	$2 \cdot 10^9$ yrs	10^{12} yrs 10^{10} yrs	~ 10 Gyrs