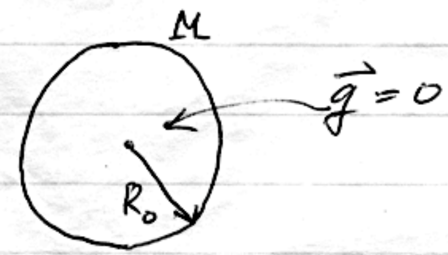
Potential of Spherical distributions

Potential of spherically symmetric mass distribution

The Poisson equation $\nabla^2 U = 4\pi G \rho$ in case of spherical system takes a simpler form:

$$\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial U}{\partial r}\right) = 4\pi G \rho(r)}$$

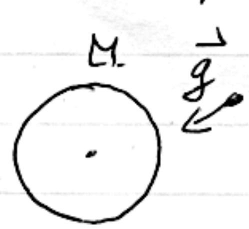
First Newton theorem ("iron sphere"): gravitational acceleration inside a spherical shell is equal to zero



We solve equation $\nabla^2 U = 0$, $r \leq R_0$ with boundary condition $U(R_0) = \mathrm{const}$

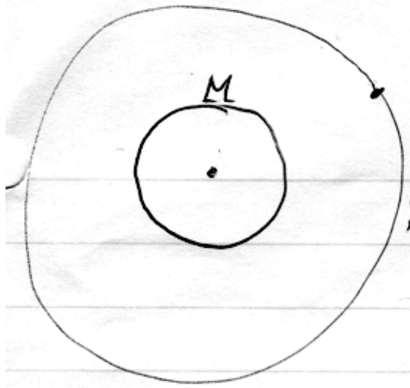
General solutions of Laplace equation $\nabla^2 U = 0$ are called the harmonic functions. Important property of harmonic functions is they can have extrema (either minimum or maximum) only on the boundary. For spherically symmetric distribution the potential on the boundary is constant. Thus, minimum of U is equal to its maximum $\Rightarrow$ potential is constant inside the shell

Second Newton theorem: For a spherical object gravitational acceleration outside of the object is the same as for point mass.



Use the Gauss law:

$$\oint \vec{g}\, d\vec{s} = -4\pi G M$$



The surface S is a sphere of radius r with center of the central object.

In this case $\oint_S \vec{g}\,d\vec{s} = -g\int d\Omega \cdot r^2 = -4\pi g r^2$

Thus, the gauss law gives

$$g = \frac{GM}{r^2}$$

Arbitrary distribution of mass in spherically symmetric system

$$\nabla^2 U = 4\pi G\rho(r) \Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) = 4\pi G\rho(r)$$

Integrate it once in limits $r \Rightarrow 0 - r$

$$r^2\frac{\partial U}{\partial r}\Big|_0^r = 4\pi G\int_0^r \rho(r)r^2 dr \equiv GM(r),$$

$M(r)$ is the mass inside r

assuming that $r^2\frac{\partial U}{\partial r}\Big|_{r=0} = 0$, we get

$$\boxed{\frac{\partial U}{\partial r} = \frac{GM(r)}{r^2}}$$

Integrate it again from 0 to r:

$$U(r) - U(0) = \int_0^r \frac{GM(r)}{r^2}dr$$, where $U(0)$ is the potential at the center of the system.

By convention, $U(\infty) = 0 \Rightarrow$ $\left\| U(0) = -\int_0^\infty \frac{GM(r)}{r^2}dr \right\|$

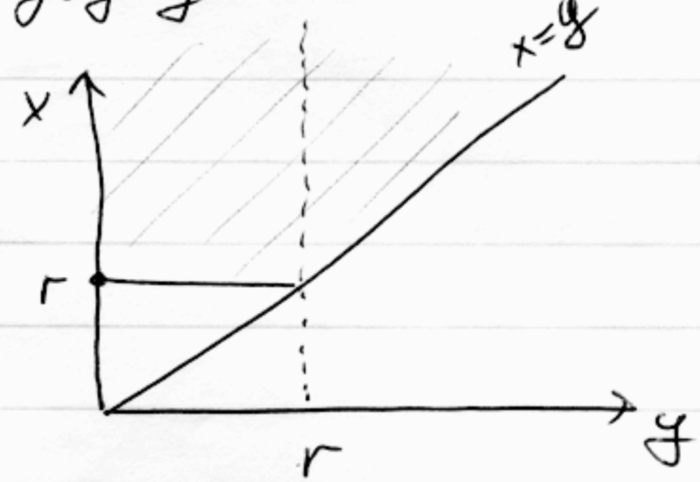
Thus,

$$\boxed{U(r) = -\int_r^\infty \frac{GM(r)}{r^2}dr}$$

Another form of $U(r)$

$$U(r) = -\int_r^\infty \frac{GM(x)}{x^2}\,dx = -\int_r^\infty \frac{G\,dx}{x^2}\int_0^x 4\pi\rho(y)y^2\,dy$$

Domain of integration:



change of the order of integration:

$$U(r) = -\int_0^r dy\cdot 4\pi\rho(y)y^2\int_r^\infty \frac{G\,dx}{x^2} -$$

$$- \int_r^\infty dy\, 4\pi\rho(y)y^2\int_y^\infty \frac{G\,dx}{x^2} = -\frac{GM(r)}{r} - \int_r^\infty \frac{G\,dm}{y},$$

here $dm = 4\pi\rho(y)y^2dy$

Examples

① Homogeneous sphere:

$$\rho = \begin{cases} \rho_0, & \text{if } r \le R \\ 0, & \text{if } r > R \end{cases} \Rightarrow M(r) = \begin{cases} \frac{4\pi}{3}\rho_0 r^3, & \text{if } r \le R \\ \frac{4\pi}{3}\rho_0 R^3, & \text{if } r > R \end{cases}$$

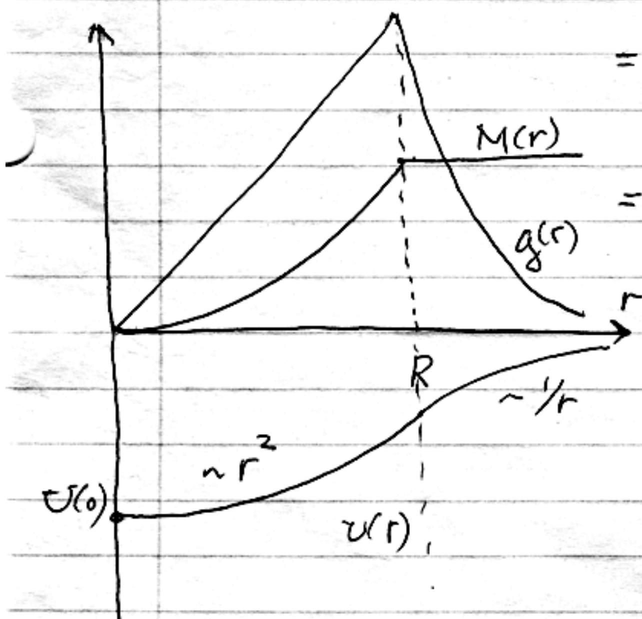
$$U(r) = -\frac{GM(R)}{r} \quad \text{for } r \ge R$$

for $r \le R$ $\quad U(r) = -\frac{GM(r)}{r} - \int_r^{\infty} \frac{G\,dm}{y} =$

$$= -\frac{4\pi}{3} G\rho_0 r^2 - \int_r^R G\,4\pi\rho_0\, y\,dy =$$

$$= -2\pi G\rho_0 \left(R^2 - \frac{r^2}{3}\right)$$

$$U(0) = -2\pi G\rho_0 R^2 = -\frac{3}{2}\frac{GM}{R}$$



King profile: $\rho = \rho_0 / \left(1 + \left(\frac{r}{a}\right)^2\right)^{3/2}$, a = core radius

$$M = 4\pi a^3 \rho_0 \left\{ \ln\left(x + \sqrt{x^2+1}\right) - x\,(x^2+1)^{-1/2} \right\}, \quad x \equiv r/a$$

$$U = -\frac{GM(r)}{r} - \frac{4\pi G\rho_0 a^2}{\sqrt{1+x^2}}$$