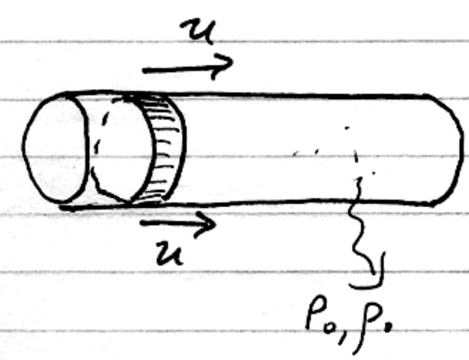
Shock Waves

Shock waves

Problem: a piston moves in to a tube filled by gas. Velocity of the piston u is small as compared with the sound velocity of gas. How gas moves in the tube?

Parameters of the gas before the piston starts to move:

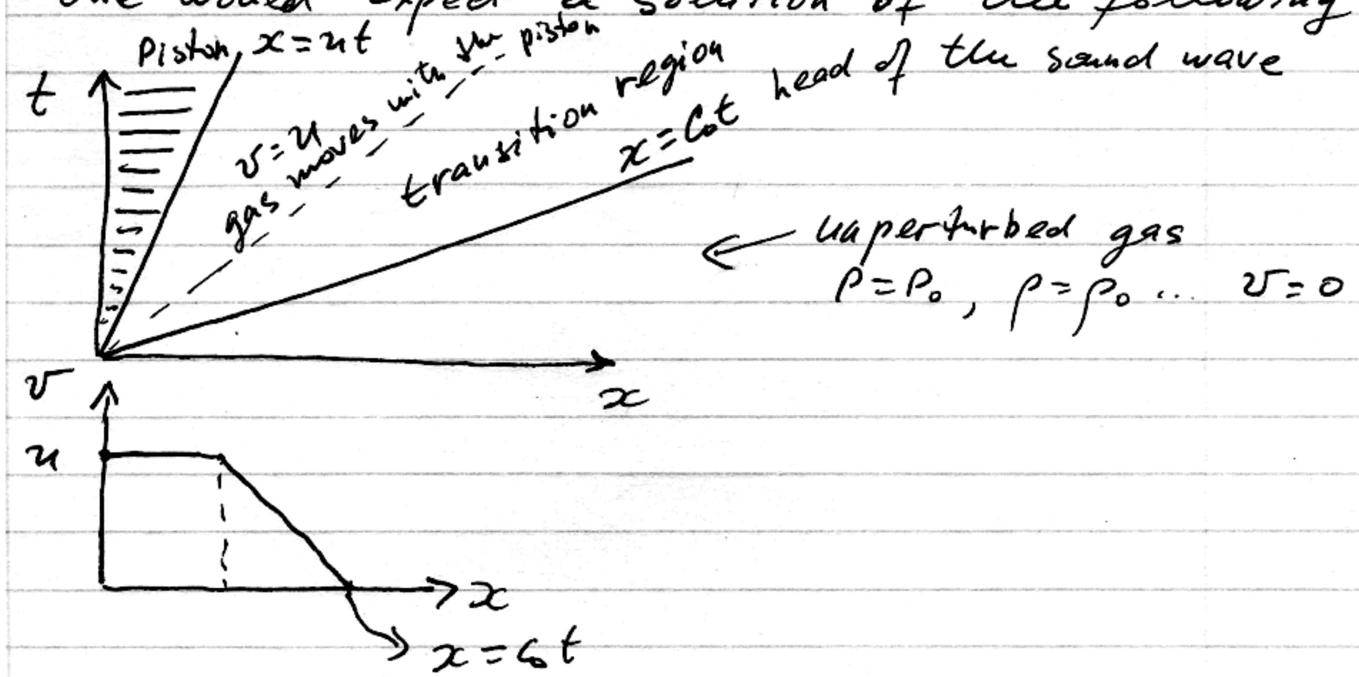


$$P = \text{const} = P_0$$

$$\rho = \rho_0, \quad v = 0, \quad c_0 = \text{const}.$$

$$u \ll c$$

One would expect a solution of the following type



In the transition region gas is slightly compressed. As the result of this compression, gas sound speed is not constant.

$$c^2 = \frac{\Delta P}{\Delta \rho} = A \cdot \gamma \rho^{\gamma - 1}.$$

Here we assume that gas is adiabatic:

$$P = A\rho^{\gamma}, \quad A = \text{const} \ (\text{"entropy"})$$

Expand in Taylor series:

$$\rho^{(\gamma-1)} = (\rho_0 + \Delta\rho)^{\gamma-1} = \rho_0^{\gamma-1}\left(1 + \frac{\Delta\rho}{\rho_0}\right)^{\gamma-1} \approx \rho_0^{\gamma-1}\left(1 + \frac{v}{c}\right)^{\gamma-1} \approx$$

$$\approx \rho_0^{(\gamma-1)}\left(1 + (\gamma-1)\frac{v}{c_0}\right)$$

v = velocity of gas

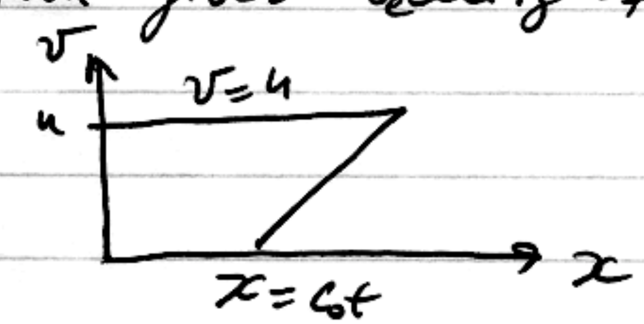
$$\frac{\Delta\rho}{\rho_0} = \frac{v}{c_0}$$

Thus,

$$c^2 = c_0^2\left(1 + (\gamma-1)\frac{v}{c_0}\right) \Rightarrow \boxed{c > c_0}$$

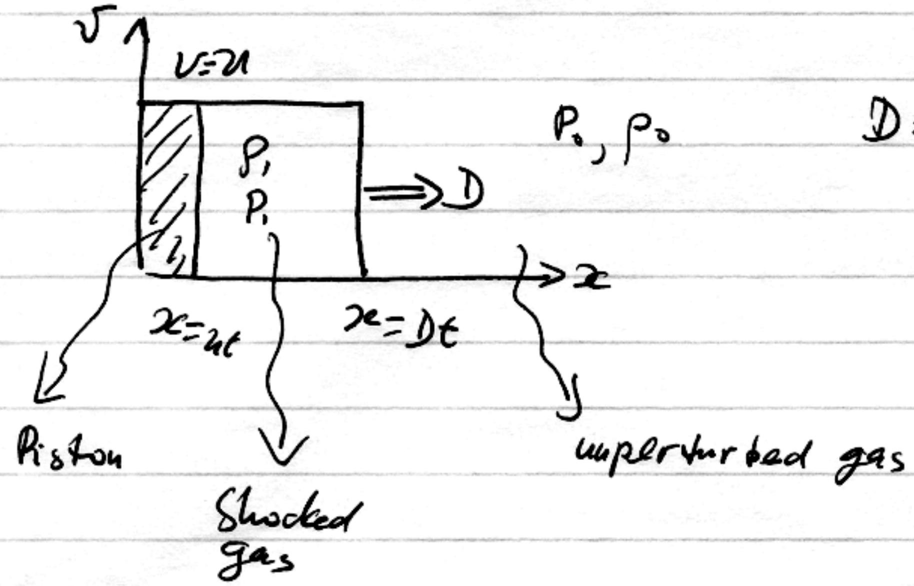
Perturbations behind the head sound wave move faster than the wave. Formal solution gives velocity of the gas in the transition region:

$$v = \frac{2}{\gamma-1}\left(\frac{x}{t} - c_0\right)$$



It is impossible to have a continuous solution

Real solution: jump conditions



D ≡ velocity of the shock

⇒ Use conservation of mass, momentum, and energy to find equations of propogation of the shock

mass: $\Delta m = 0$ $\quad \rho_1 (D-u)\cdot t = \rho_0 D t$

momentum: $\Delta m v = f \Delta t$ $\quad \rho_0 D u t = (P_1 - P_0) t$

energy: change of the total energy = work of external force to move the piston

$$\rho_0 D t \left(\varepsilon_1 - \varepsilon_0 + \frac{u^2}{2}\right) = P_1 u t \; ; \quad \varepsilon = \text{internal energy per unit mass}$$

The equations can be rewritten in the following form:

$$\begin{cases} \rho_1 (D-u) = \rho_0 D \\ \rho_0 D u = P_1 - P_0 \\ \rho_0 D\left(\Delta\varepsilon + \frac{u^2}{2}\right) = P_1 u \end{cases}$$

+ equation of state

For ideal gas $\qquad \varepsilon = \frac{1}{\gamma - 1} \frac{P}{\rho}$

we can find that $\qquad D\left[D - \left(1 + \frac{\sigma}{2}\right)u_1\right] = \frac{\gamma P_0}{\rho_0} = c_0^2$

Thus $\boxed{D > c_0}$

<u>Another reference frame</u>: shock is not moving

$u_1 \leftarrow \quad | \quad \leftarrow u_0$

$P_1, \rho_1, c_1 \quad | \quad P_0, \rho_0, c_0$

mass flux: $\boxed{\rho_0 u_0 = \rho_1 u_1}$

momentum of mass $\rho_0 u_0$ is $\rho_0 u_0 \cdot u_0$

The increase of the momentum after passing through the shock is $\rho_1 u_1^2 - \rho_0 u_0^2$. It is equal to the impulse produced by the pressure force $P_0 - P_1$:

$$\boxed{\rho_1 u_1^2 + P_1 = \rho_0 u_0^2 + P_0}$$

In other words, momentum fluxes are equal on both sides of the shock.

The difference in energies

$$\rho_0 u_0 \left(\left(\varepsilon_1 + \frac{u_1^2}{2}\right) - \left(\varepsilon_0 + \frac{u_0^2}{2}\right) \right) = \text{work done by pressure} =$$

$$= P_0 u_0 - P_1 u_1$$

$$\boxed{\varepsilon_1 + \frac{u_1^2}{2} + \frac{P_1}{\rho_1} = \varepsilon_0 + \frac{u_0^2}{2} + \frac{P_0}{\rho_0}}$$

Hugoniot curves

introduce specific volume $V = 1/\rho$

mass conservation gives $\frac{V_0}{V_1} = \frac{u_0}{u_1}$

momentum flux: $p_1 + \rho_1 u_1^2 = p_0 + \rho_0 u_0^2 \Rightarrow \frac{u_1^2}{V_1} - \frac{u_0^2}{V_0} = p_0 - p_1 \Rightarrow$

$$\Rightarrow u_1^2 = V_1^2 \frac{(p_1 - p_0)}{V_0 - V_1} \;;\quad u_0^2 = V_0^2 \frac{(p_1 - p_0)}{V_0 - V_1}$$

Substituting these expressions into energy equation:

$$(*) \qquad \varepsilon_1 - \varepsilon_0 = \frac{1}{2}(p_1 + p_0)(V_0 - V_1)$$

Because $\varepsilon = \varepsilon(p, V)$ equation $(*)$ gives a relation between p_1 and three other parameters of the flow: V_1, p_0, V_0

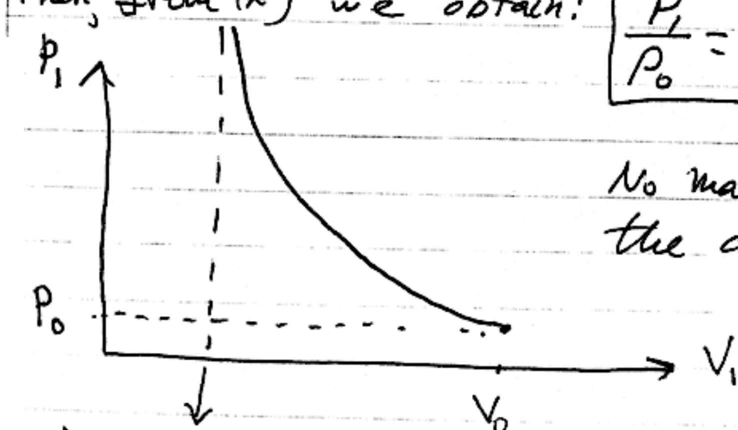
$$\boxed{p_1 = p_1(V_1, p_0, V_0)}$$ This is called Hugoniot curve

(which is analogous to usual adiabate: $p = p(V)$ or $p = p(\rho)$)

Ideal gas

for ideal gas $\varepsilon = \frac{pV}{\gamma - 1} = c_V T$

Then, from $(*)$ we obtain: $\boxed{\frac{p_1}{p_0} = \frac{(\gamma+1)V_0 - (\gamma-1)V_1}{(\gamma+1)V_1 - (\gamma-1)V_0}}$ Hugoniot curve for ideal gas



$$\boxed{\frac{V_1}{V_0} = \frac{\gamma-1}{\gamma+1}}$$

No matter how strong is the shock, the compression of gas $\frac{\rho_1}{\rho_0}$ cannot be larger than

$$\boxed{\frac{\rho_1}{\rho_0} = \frac{\gamma+1}{\gamma-1}}$$

From $P_1, P_0, \rho_0; V_0$ we can find temperature T:

$$\boxed{\frac{T_1}{T_0} = \frac{P_1 V_1}{P_0 V_0}}$$

Strong shock:

$P_1 \gg P_0$

$$\frac{\rho_1}{\rho_0} = \frac{\gamma+1}{\gamma-1} \quad \begin{aligned} &= 4 \text{ for } \gamma = 5/3 \\ &= 6 \text{ for } \gamma = 7/5 \text{ (diatomic gas)} \end{aligned}$$

$$\frac{T_1}{T_0} = \frac{\gamma-1}{\gamma+1} \frac{P_1}{P_0}$$

Velocities:

$$u_1^2 = V_1^2 \frac{(P_1 - P_0)}{V_0 - V_1} \; ; \quad c_1^2 = \gamma P_1 V_1$$

$$u_0^2 = V_0^2 \frac{(P_1 - P_0)}{V_0 - V_1} \; ; \quad c_0^2 = \gamma P_0 V_0$$

$$\left(\frac{u_0}{c_0}\right)^2 = \frac{(\gamma-1) + (\gamma+1) P_1/P_0}{2\gamma} \longrightarrow \frac{(\gamma+1)}{2\gamma} \cdot \frac{P_1}{P_0} \gg 1$$

$$\left(\frac{u_1}{c_1}\right)^2 = \frac{(\gamma-1) + (\gamma+1) P_0/P_1}{2\gamma} \rightarrow \frac{\gamma-1}{2\gamma} \leq \frac{1}{2} < 1$$

Similarity solutions and propagation of a strong spherical shock (Sedov solution)

General solutions of hydro equations should give physical properties of gas:

$$P = P(x,t)\,;\ \rho = \rho(x,t)\,;\ v = v(x,t),$$

where x and t are coordinates and time

Pressure P and density ρ have dimensionality of mass

$$[P] = g\ cm^{-1} sec^{-2} \qquad [\rho] = g\ cm^{-3}$$

at the same time x and t do not have gram dimensionality. This means that some parameter with [gram] should be present. It can come from either boundary or initial conditions. For example, initial unperturbed density ρ_0 may serve as the parameter.

In general, solutions of hydro equation have the form:

$$P = \frac{a}{x_0^{k+1} t_0^{s+2}} \mathbb{P}\left(\frac{x}{x_0}, \frac{t}{t_0}\right);\ \rho = \frac{a}{x_0^{k+3} t_0^{s}} \mathbb{G}\left(\frac{x}{x_0}, \frac{t}{t_0}\right)$$

$$v = \frac{x_0}{t_0} \mathbb{V}\left(\frac{x}{x_0}, \frac{t}{t_0}\right)$$

Here parameter a has dimensions $[a] = g \cdot cm^{k} sec^{s}$

Functions $\mathbb{P}$, $\mathbb{G}$, $\mathbb{V}$ are dimensionless functions

x_0 and t_0 are dimensional parameters

If parameters of the system can't produce a combination with length dimension and a combination with time dimension, variable x and t cannot enter functions $\mathbb{P}$, $\mathbb{G}$, and $\mathbb{V}$ separately – only in a combination of x and t

This indicates that the system has a similar solution (or self-similar solution).

Now $\mathbb{P}$, $\mathbb{G}$, and $\mathbb{V}$ are functions of one dimensionless argument, ξ

$$\boxed{\xi = \frac{x}{A t^{\alpha}}}$$

A = dimensional parameter

[$A = c_0$ for small ampl. wave: $\frac{\Delta\rho}{\rho} = \frac{v}{c_0} = f_1(x - c_0 t)$

$\alpha = 1$ in this case]

now

$$p = \frac{a}{x^{k+1} t^{s+2}} \mathbb{P}(\xi)$$

$$\rho = \frac{a}{x^{k+3} t^{s}} \mathbb{G}(\xi), \quad v = \frac{x}{t} \mathbb{V}(\xi)$$

Because there is only one independent variable ξ, equations of hydrodynamics, which were eqs in partial derivatives, become a system of ordinary differential equations. It is much easier to deal with ODE's.

Important example: strong point explosion in homogeneous atmosphere

Sedov, Staniukovich, (1955)
(1946) Taylor (1950)

initial conditions: $\rho = \rho_0 = \text{const}$

E_0 is released at $r = 0$, $t = 0$

pressure p_0 is much smaller, than the pressure at the shock $\Rightarrow$ initial pressure is neglected: $p_0 \simeq 0$

the only parameters of the problem are

$$E = \text{energy of the explosion} \quad [E] = g\,\frac{cm^2}{sec^2}$$

$$\rho_0 = \text{initial density} \quad [\rho_0] = \frac{g}{cm^3}$$

Using E and ρ_0 we cannot get a combination with dimension sec or cm - only

$$[E/\rho_0] = cm^5 sec^{-2} \Rightarrow \left[\frac{E}{\rho_0}\right]^{1/5} = \frac{cm}{sec^{2/5}} \Rightarrow \alpha = \frac{2}{5}$$

This means that we have self-similar solution and the variable ξ is

$$\boxed{\xi = r\left(\frac{\rho_0}{E t^2}\right)^{1/5}}$$

Position of the shock wave corresponds to some value ξ, say ξ_0

Thus, the shock position is

$$\boxed{R_{shock} = \xi_0 \cdot \left(\frac{E}{\rho_0}\right)^{1/5} t^{2/5}}$$

Velocity of the shock

$$D = \frac{dR_{shock}}{dt} = \frac{2}{5}\frac{R_{shock}}{t} = \frac{2}{5}\xi_0^{5/2}\left(\frac{E}{\rho_0}\right)^{1/2} R^{-3/2}$$

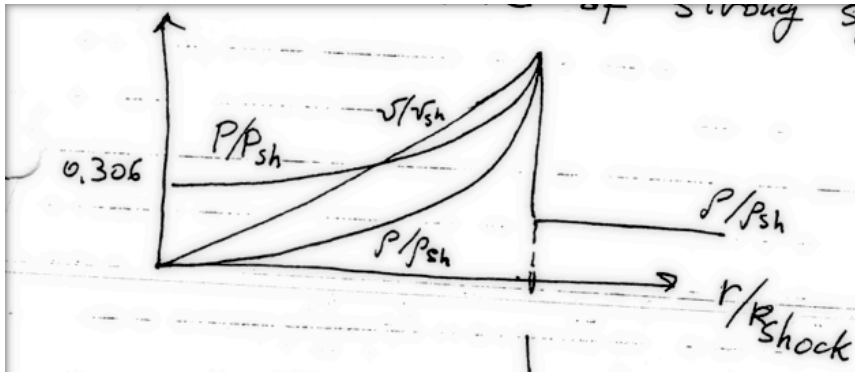
parameters at the shock are given by strong-shock approximation:

$$\frac{\rho_1}{\rho_0} = \frac{\gamma+1}{\gamma-1}, \quad P_1 = \frac{2}{\gamma+1}\rho_0 D^2, \quad u_1 = \frac{2}{\gamma+1} D$$

Parameter ξ_0 is defined from the condition that initial energy E is now spread over all volume inside the shock:

$$E = \int_0^{R_{shock}} 4\pi r^2 \rho \cdot \left(\varepsilon + \frac{u^2}{2}\right) dr$$

... of strong spherical shock:



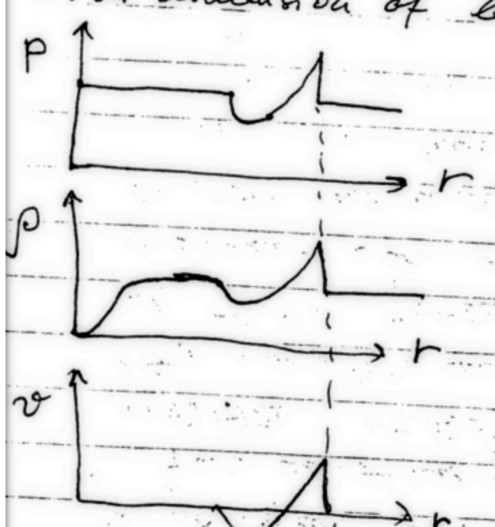
$\gamma = 5/3$

$\Rightarrow$ Later stages of evolution of a strong shock:

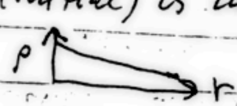
$$\left.\begin{array}{l} P_{sh} \sim \rho_0 D^2 \sim \dfrac{E}{R^3} \\ D \sim R^{-3/2} \end{array}\right\} \Rightarrow$$ shock gets weaker as it goes from the center (because it spreads its energy over larger mass)

At some moment $D \sim c_0$ and shock becomes usual acoustic wave

Note that the solution at this stage is not self-similar because we cannot neglect the pressure of unperturbed gas. This pressure p_0 in combination with E gives a parameter with dimension of length: $r_0 = \left(\frac{E}{p_0}\right)^{1/3}$



If density (initial) is not constant:

if  shock accelerates

if  shock decelerates

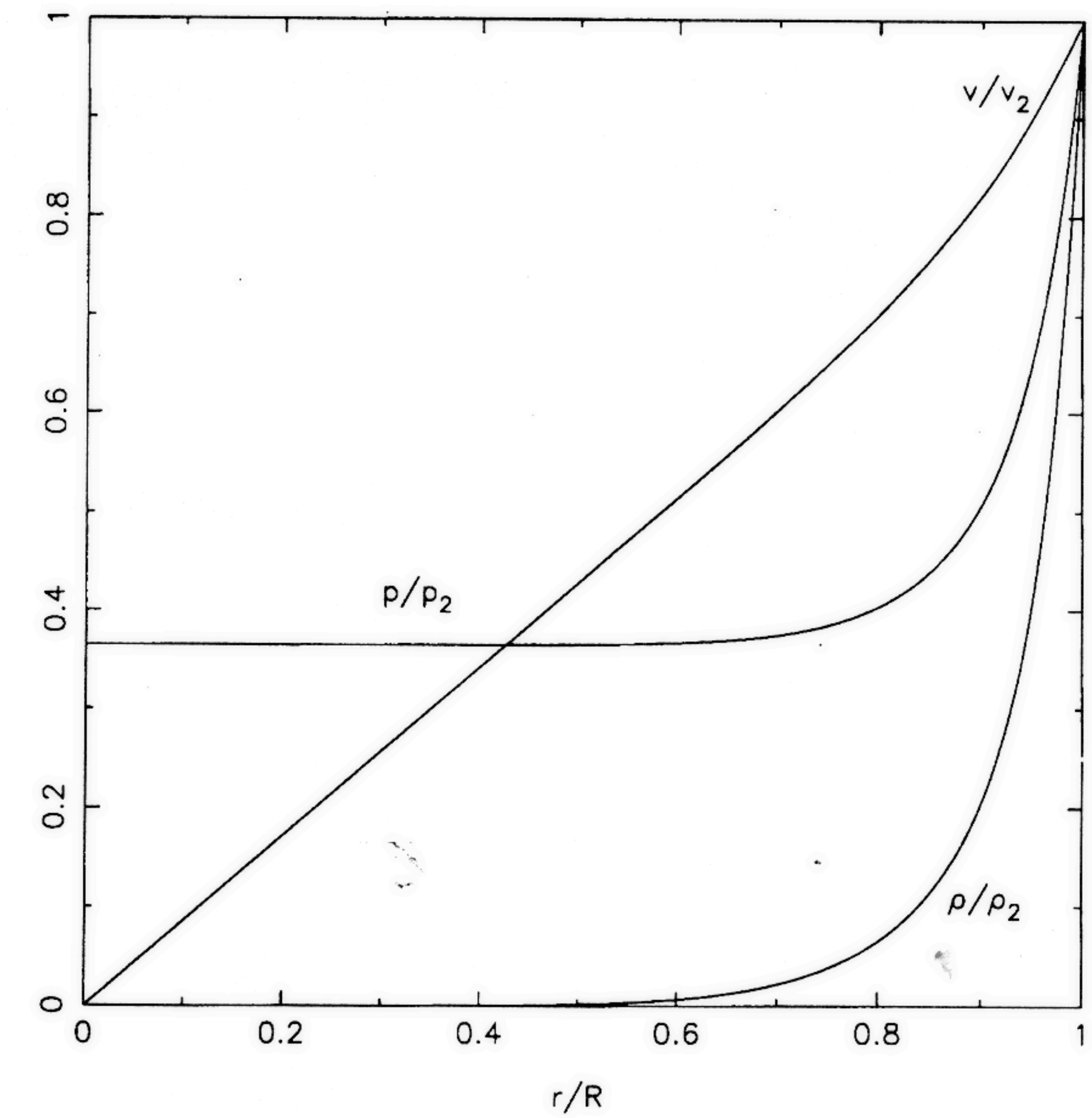
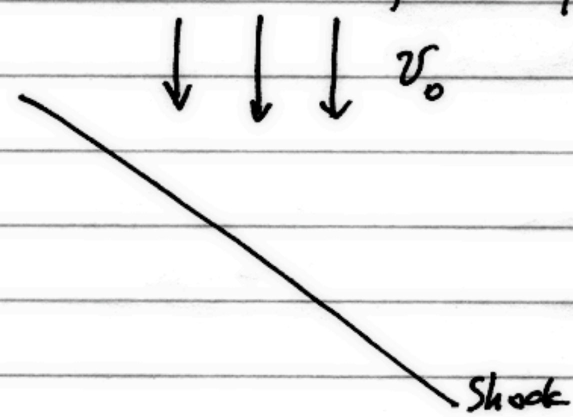
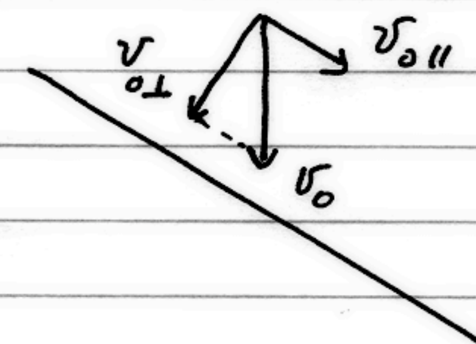


Fig. 3.9 The velocity, pressure and density for the Sedov solution.

Oblique shock

Consider a shock wave, which propages at some angle relative to the flow of gas:





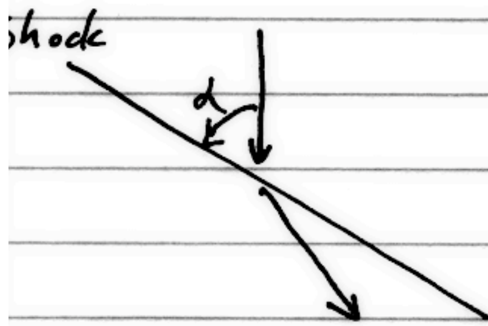
Perpendicular component of velocity $v_{0\perp}$ will decrease after the shock:

$$\rho_1 v_{1\perp} = \rho_0 v_{0\perp}$$

Because $\rho_1 > \rho_0 \Rightarrow v_{1\perp} < v_{0\perp}$

However, the parallel component is preserved: $v_{1\parallel} = v_{0\parallel}$

As the result, streamlines get closer to the shock:



Gas before the shock must be supersonic. After the shock gas can be either sub or super-sonic.

There is a limit on angle α, which follows from condition that $v_{0\perp} > c_0$

From geometry $v_{0\perp} = v_0 \sin\alpha \Rightarrow$

$$\sin\alpha > \frac{c_0}{v_0} \equiv \frac{1}{M_0}$$