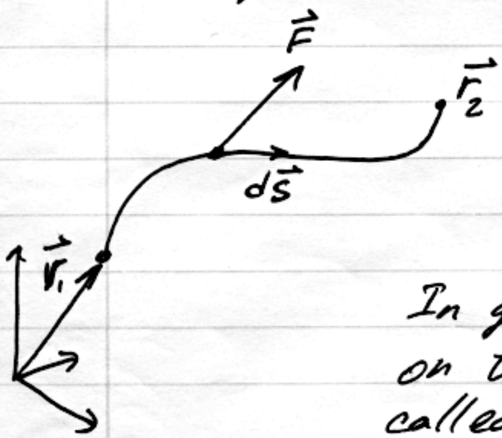
# Gravitational Potential

# Gravitational potential:

- work done by the force of gravity $\vec{F}(\vec{r})$ to displace a particle of unit mass from position $\vec{r}_1$ to position $\vec{r}_2$:




$$W_{12} = \int_1^2 \vec{F} d\vec{s} = \int_1^2 (F_x ds_x + F_y ds_y + F_z ds_z)$$

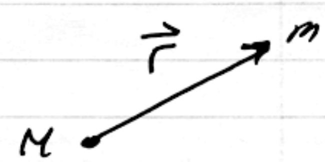
In general, if the work $W_{12}$ does not depend on the path of integration, the force is called conservative or potential. In this case, $W_{12}$ is the same for any path joining positions $\vec{r}_1$ and $\vec{r}_2$

Definition: force $\vec{F}$ is central if it depends only on radius-vector $\vec{r}$:

$$\vec{F}(\vec{r}) = \varphi(r)\vec{r}$$

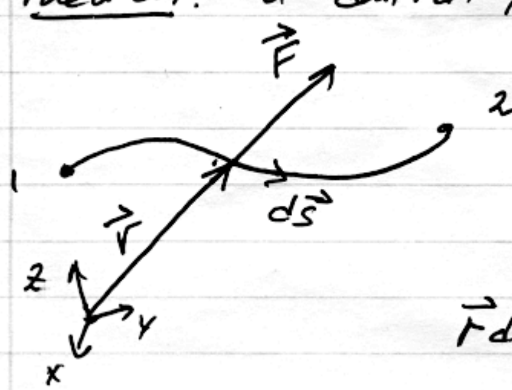
Newtonian force

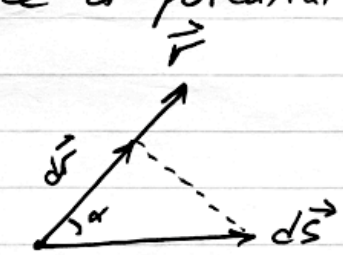
$$\vec{F} = -\frac{GMm}{|r^3|} \cdot \vec{r}$$




is the central force

Theorem: A central force is potential force




$\vec{r} d\vec{s} = r \cos\alpha \cdot ds = r dr$, where $dr$ is the projection of $d\vec{s}$ on $\vec{r}$

By definition $\vec{F} = \Phi(r)\vec{r}$

The work $W_{12}$ can be written as

$$W_{12} = \int_1^2 \vec{F} d\vec{s} = \int_1^2 \Phi(r)\vec{r}\cdot d\vec{s} = \int_{r_1}^{r_2} \Phi(r) r\, dr$$

The last integral depends only on $r_1$ and $r_2$, not on a particular path joining 1 and 2. Thus, the force is potential.

We can define gravitational potential as the work done by the force to move a particle of unit mass from $\vec{r}$ to infinity. This is convinient if the system is finite (does not extend to $\infty$)

$$(*) \qquad U(\vec{r}) = \int_{\vec{r}}^{\infty} \vec{F} d\vec{r} = \int_{\vec{r}}^{\infty} F_x dx + F_y dy + F_z dz$$

If we differentiate $(*)$ in regard to $\vec{r}$, we get inverse relation:

$$(**) \qquad \vec{F} = -\operatorname{grad} U(r) \equiv -\vec{\nabla} U(r)$$

In particular case of the central force $(**)$ can be written as

$$(***) \qquad \vec{F} = -\frac{dU(r)}{dr}\frac{\vec{r}}{|r|}$$

For a particle of unit mass eq$(***)$ means that the trajectory of the particle is

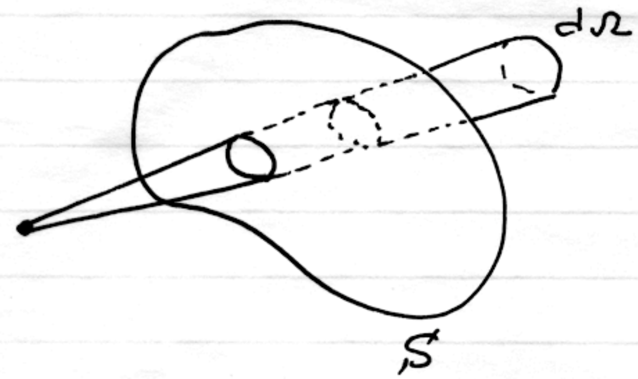
$$(****) \qquad \ddot{\vec{r}} \equiv \frac{d^2\vec{r}}{dt^2} = -\frac{dU}{dr}\frac{\vec{r}}{|\vec{r}|}$$

It is convinient to define $U(r)$ as work per unit mass. Then $(****)$ is valid for any particle.

# Gravitational potential: general results

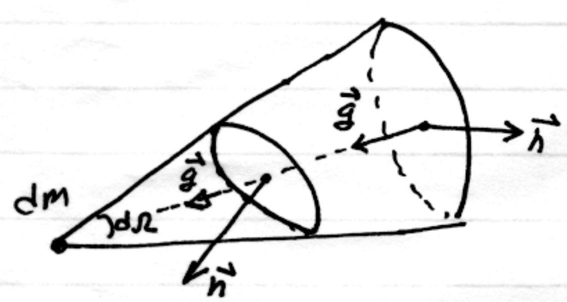
(I) The Poisson equation. The goal is to write a differential equation relating density $\rho(\vec{r})$ and gravitational potential $\varphi(\vec{r})$ [or $U(\vec{r})$]

Let's start with a point mass $dm$. Find flux of $\vec{g}$ through an arbitrary closed surface, that does not include the mass $dm$.

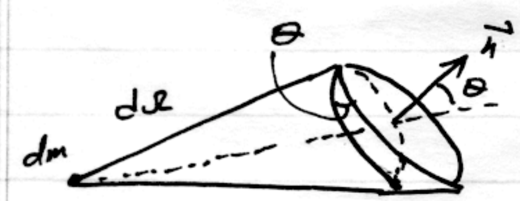



$d\Omega$ is solid angle

What is $\oint_S \vec{g} \cdot d\vec{s} = ?$




$\vec{n}$ is a unit vector orthogonal to the surface and directed away from the volume



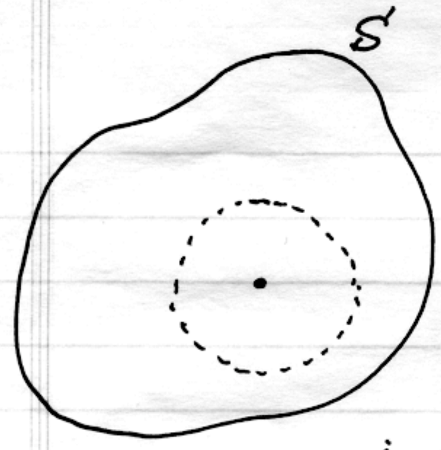

$$d\vec{s} = \frac{d\Omega\, r^2}{\cos\theta} \cdot \vec{n}$$

$$\vec{g}\, d\vec{s} = -g\, ds \cos\theta = -g r^2 d\Omega =$$

$$= -\frac{G\, dm}{\cancel{r^2}} \cancel{r^2}\, d\Omega = -G\, dm\, d\Omega$$

The signs are different for the first and the second crossing of the cone with the surface, but the absolute values of $(\vec{g}\, d\vec{s})$ are the same $\Rightarrow$ The sum is equal zero:

$$\oint \vec{g}\, d\vec{s} = 0$$




Now find the flux of $\vec{g}$ through a surface, which encompasses mass $dm$. Place an auxiliary sphere around $dm$. The total flux through surface defined by $S$ and the sphere is equal to zero because $dm$ is not surrounded by $S$+ sphere:

$$\int_{S+\text{sphere}} \vec{g}\,d\vec{s} = 0 = \int_S \vec{g}\,ds + \int_{\text{sphere}} \vec{g}\,d\vec{s}$$

at the same time the flux through the sphere is

$$\int_{\text{sphere}} \vec{g}\,d\vec{s} = \int \frac{G\,dm}{r^2}\cdot r^2 d\Omega = 4\pi G\,dm$$

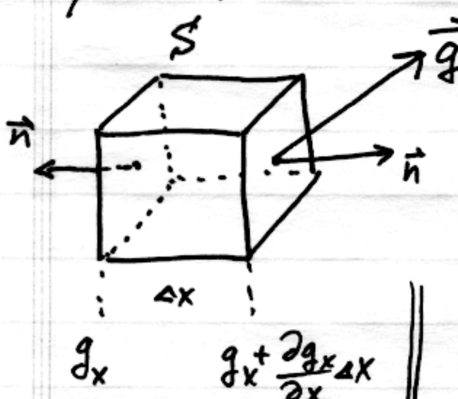
Thus, $$\int_S \vec{g}\,d\vec{s} = -4\pi G\,dm.$$

For arbitrary distribution of masses $dm_1, dm_2, dm \ldots$ the force is the sum of forces $\vec{g} = \vec{g}_1 + \vec{g}_2 + \vec{g} \ldots$

Sum up all contributions:

**Gauss Law** $$\int_S \vec{g}\cdot d\vec{s} = -4\pi G \underline{M},$$ where $M$ is the total mass inside surface $S$

We need to re-write the Gauss law as a differential equation. Place a small cube in space. Mass inside the cube is $dm = \rho\,\Delta V$ ; $\Delta S = \Delta x \Delta y = \Delta y \Delta z = \Delta x \Delta z$




$$\int_S \vec{g}\,d\vec{s} = \left(g_x + \frac{\partial g_x}{\partial x}\Delta x\right)\Delta S - g_x \Delta S + \left(g_y + \frac{\partial g_y}{\partial y}\Delta y\right)\Delta S - g_y \Delta S + (\ldots z \ldots\ldots) =$$

$$= \left[\frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y} + \frac{\partial g_z}{\partial z}\right]\Delta V = \operatorname{div}\vec{g}\,\Delta V$$

The differential form of the Gauss law is the Poisson equation:

$$\vec{\nabla}\vec{g} = -\nabla^2 U = -4\pi G\rho \quad \Big\| \text{ Here we used } \vec{g} = -\vec{\nabla} U$$

Note, that this a linear equation. If $U_1$ is a solution for $\rho_1$ and $U_2$ is solution for $\rho_2$, than

$\alpha U_1 + \beta U_2$ is a solution for $\alpha\rho_1 + \beta\rho_2$

The formal solution of the Poisson equation is (Green functions):

$$\vec{g}(\vec{r}) = -G\int_{-\infty}^{\infty} \frac{(\vec{r}-\vec{r}\,')\rho(\vec{r}\,')d^3r'}{|\vec{r}-\vec{r}\,'|^3}, \quad d^3r' = dx\,dy\,dz$$

$$U(\vec{r}) = -G\int \frac{\rho(\vec{r}\,')d^3r'}{|\vec{r}-\vec{r}\,'|}$$

<u>Potential Energy</u>: given $\rho(\vec{x})$ and $U(\vec{x})$, find the total gravitational energy of the system $W$

By definition, $U(\vec{x})$ is a potential energy of a unit mass at the position $\vec{x}$: $\quad \delta W = U(\vec{x})dm = U(\vec{x})\rho\, dV$

We cannot integrate $\delta W$ to get the total grav. energy of the system. What we need to do is to write a varience of the total energy in a form of differentials.

Let's change the density everywhere by small amount $\delta\rho(\vec{x})$. The potential energy changes by

$$\delta W = \int \delta\rho\, U(\vec{x})\, dV. \Leftarrow \text{Integral is for all space } (-\infty, +\infty)$$

On the other hand, $\delta\rho$ produces a deviation of gravitational potential $\delta U$:

$$\nabla^2 \delta U = 4\pi G \delta\rho$$

Thus, $\delta W = \frac{1}{4\pi G} \int U(x) \nabla^2 (\delta U) dV$

Integrate over space once by parts:

$$\int_{-\infty}^{\infty} U(x) \left[ \frac{\partial^2}{\partial x^2} \delta U + \frac{\partial^2 \delta U}{\partial y^2} + \frac{\partial^2 \delta U}{\partial z^2} \right] dx\, dy\, dz$$

Take the first term and integrate it over x

$$+\int_{-\infty}^{\infty} dx\, U(x) \frac{\partial^2 \delta U}{\partial x^2} = U(x) \frac{\partial}{\partial x} \delta U \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{\partial U}{\partial x} \cdot \frac{\partial \delta U}{\partial x} dx$$

The first term on the right is equal to zero ($U(\infty) = 0$)

Note that

$$\frac{\partial U}{\partial x} \cdot \frac{\partial \delta U}{\partial x} = \frac{1}{2} \delta \left[ \left( \frac{\partial U}{\partial x} \right)^2 \right]$$

Thus, the integral for $\delta W$ can be written as a full differential:

$$\delta W = -\frac{1}{4\pi G} \frac{1}{2} \delta \left[ \int dv \left\{ \left( \frac{\partial U}{\partial x} \right)^2 + \left( \frac{\partial U}{\partial y} \right)^2 + \left( \frac{\partial U}{\partial z} \right)^2 \right\} \right]$$

In vector form:

$$W = -\frac{1}{8\pi G} \int dv\, |\vec{\nabla} U|^2$$

Another form of this expression involves $U$ and $\rho$.

Integrate by parts: $\int \left( \frac{\partial U}{\partial x} \right)^2 dx = -\int U \frac{\partial^2 U}{\partial x^2} dx$

Then use $\nabla^2 U = 4\pi G \rho \Rightarrow$

$$\boxed{W = \frac{1}{2} \int \rho U dv}$$

Note the factor $\frac{1}{2}$