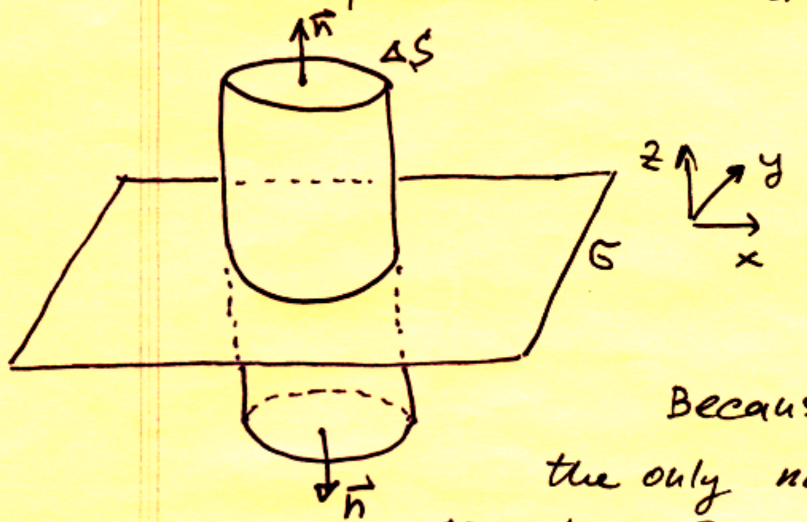
Potentials for Non-spherical Distributions

Non-spherical systems: examples, which demonstrate general trends

In general, non-spherical systems are much more difficult to handle. Still, we can solve some simple and astronomically interesting cases

(I) Thin infinite disk: a simple model for stellar disks



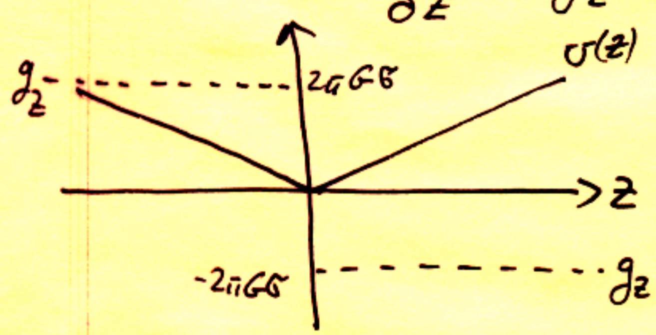
σ is the surface density
ΔS = area
$\vec{n}$ = unit vector orthogonal to the surface

Because of the plane symmetry, the only non-zero component of $\vec{g}$ is g_z

Use the Gauss law to find g_z:

$$\oint_{\text{cylinder}} \vec{g}\, d\vec{S} = -4\pi G M \Rightarrow \left.\begin{array}{l} M = \sigma \Delta S \\ \oint \vec{g}\, ds = -2 g_z \Delta S \end{array}\right\} \rightarrow \vec{g} = -2\pi G \sigma \vec{n}$$

Now, from $\frac{\partial U}{\partial z} = -g_z$ find $U = 2\pi G \sigma |z|$



If we start with the Poisson equation, we get the same answer. In this case $\rho = \sigma \delta(z)$

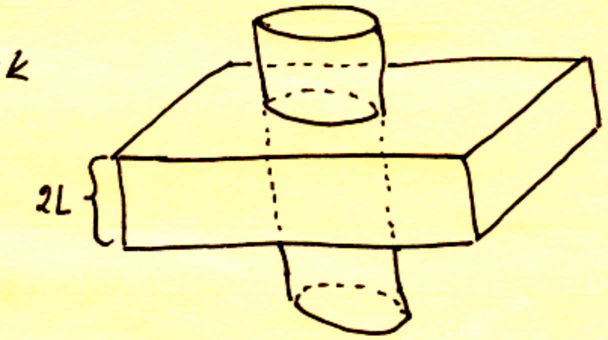
Note, that in this case we cannot normalize U in usual way to have $U(\infty) = 0$. Instead, we use $U(0) = 0$.
Another effect: in spite of infinite density at $z=0$, both g_z and U are finite.

(II) Slightly more complicated system: Thick disk

$\rho_0 = const$ = density of the disk

$2L$ = disk height

$\sigma \equiv 2\rho_0 L$ = surface density



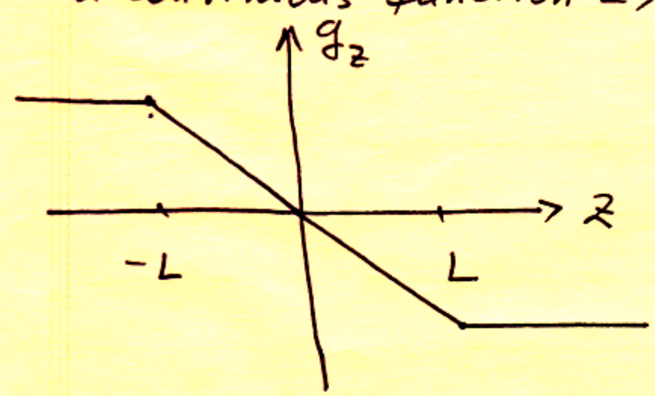
$M(z)$ = mass inside the cylinder within $|z|$

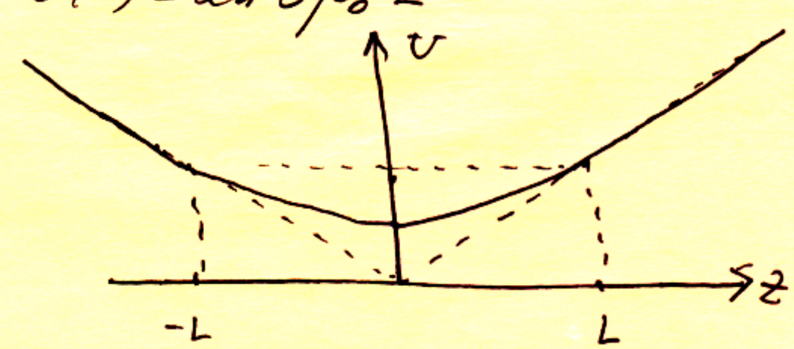
$$M(z) = \begin{cases} \sigma \Delta S, & \text{for } |z| \geq L \\ 2\rho_0 |z| \Delta S, & \text{for } |z| < L \end{cases}$$

$$\oint \vec{g}\, d\vec{s} = -4\pi G M \Rightarrow g(z) = \begin{cases} -4\pi G \rho_0 L, & z \geq L \\ -4\pi G \rho_0 z, & 0 < z < L \end{cases}$$

$$U = \begin{cases} 4\pi G \rho_0 L z, & z \geq L \\ 2\pi G \rho_0 z^2 + U(0), & z < L \end{cases}$$

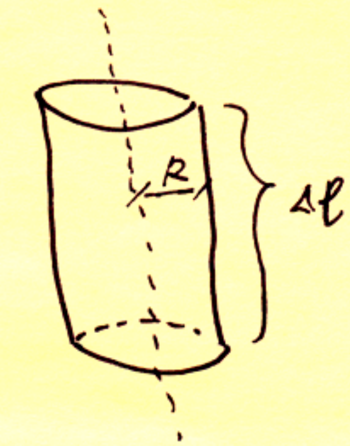
The constant should be chosen in such a way, that U is a continious function $\Rightarrow U(0) = 2\pi G \rho_0 L^2$





$U(L) = 2U(0)$

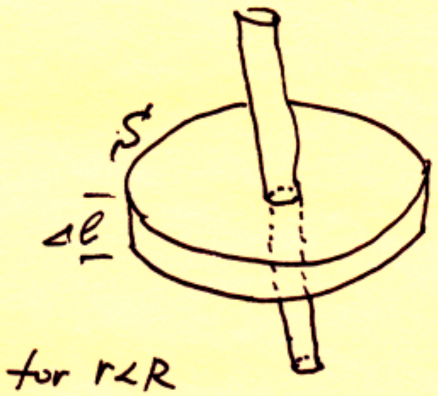
III Filament with constant density



The filament is infinite; radius is R; density is ρ_0

$\sigma(r)$ = mass per unit length inside radius r

$$\sigma(r) = \pi\rho_0 \begin{cases} r^2, & \text{for } r < R \\ R^2, & \text{for } r \geq R \end{cases}$$



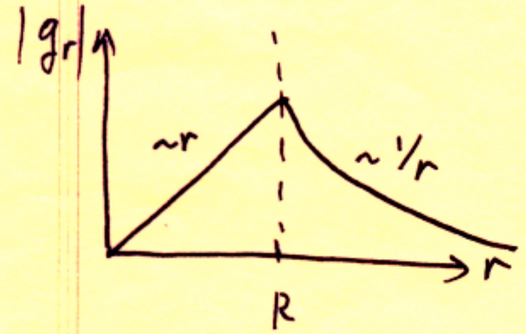
Apply the Gauss law: $\oint \vec{g}\,d\vec{s} = -4\pi G M$

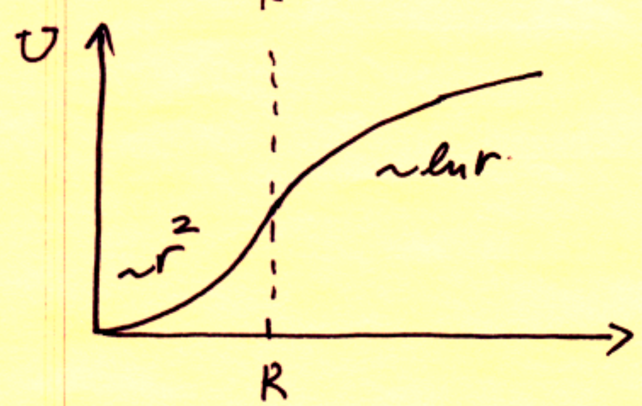
$$g_r \Delta l \, 2\pi r = -4\pi G \sigma(r) \Delta l$$

This gives:

$$g_r = -\frac{2G\sigma(r)}{r} = \begin{cases} -2\pi G\rho_0 r, & \text{for } r<R \\ -2\pi G\rho_0 \dfrac{R^2}{r}, & r \geq R \end{cases}$$

$$U(r) = \begin{cases} \pi G\rho_0 r^2 + U(0), & \text{for } r<R \\ \pi G\rho_0 R^2 + 2\pi G\rho_0 R^2 \ln\left(\dfrac{r}{R}\right) + U(0); & r \geq R \end{cases}$$





We can find the potential for a string with finite mass by setting $R \to 0$ and keeping $\sigma = \pi\rho_0 R^2 = \text{const} = \sigma_0$

In this case

$$U = 2\sigma_0 G \ln r$$

This goes to ∞ both at $r \to 0$ and at $r \to \infty$

Homogeneous Ellipsoid: $\rho = \rho_0$ is constant inside ellipsoid:

$$\frac{x^2}{a^2} + \frac{b^2}{c^2} + \frac{z^2}{c^2} = 1$$

Where a, b.a are semi axes

$$U(x,y,z) = \pi G \rho_0 \left[A x^2 + B y^2 + C z^2 \right]$$

Gravitational potential inside the ellipsoid is given by this expression.

Here coefficient A, B, and C are:

$$A = abc \int_0^\infty \frac{d\lambda}{(a^2+\lambda) f^{1/2}},$$

$$B = abc \int_0^\infty \frac{d\lambda}{(b^2+\lambda) f^{1/2}}, \quad C = abc \int_0^\infty \frac{d\lambda}{(c^2+\lambda) f^{1/2}}; \; f \equiv (a^2+\lambda)(b^2+\lambda)(c^2+\lambda)$$

~~analysis of these relations.~~

Use the Poisson equation

$$\nabla^2 U = 4\pi G \rho(x,y,z)$$

This gives the relation between ff, B, c

$$A + B + C = 2$$

For an oblate ellipsoid the coefficients A, B, and C are

$$a = b > c$$

$$A = B = \frac{(1-e^2)^{1/2}}{e^2} \left[\frac{\arcsin e}{e} - (1-e^2)^{1/2} \right], \quad \text{where } e^2 = 1 - \frac{c^2}{a^2}$$

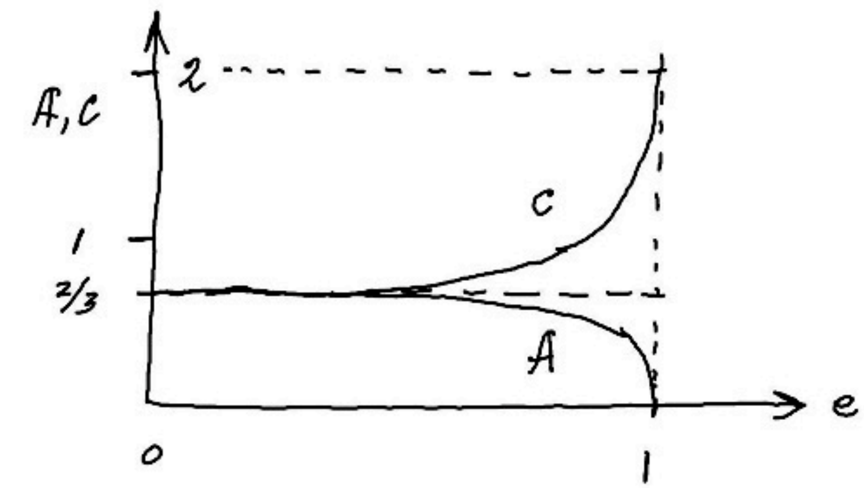
$$C = 2 \frac{(1-e^2)^{1/2}}{e^2} \left[\frac{1}{(1-e^2)^{1/2}} - \frac{\arcsin e}{e} \right], \quad e = \text{eccentricity}$$

at small $e \ll 1$ $\quad \arcsin e \approx e + \frac{e^3}{3!} \Rightarrow A = B \simeq \frac{2}{3}$

at $e = 1$ $\quad \arcsin(1) = \pi/2, \quad A \simeq \frac{\pi}{2}\sqrt{1-e^2} = \frac{c}{a}\frac{\pi}{2} \to 0$

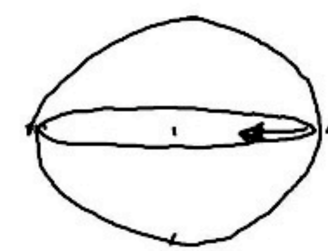
$$C \simeq 2$$

so, in this limit $(e \simeq 1, a \gg c)$ $\quad U \simeq 2\pi G \rho_0 z^2$



How much the acceleration changes if we flatten the distribution?

$$\left.\frac{\partial U}{\partial x}\right|_{\substack{x=a\\ y=0\\ z=0}} = 2\pi G\rho_0 A x$$



Rewrite it in a different way:

Mass is equal to

$$M = \frac{4\pi}{3} a b c \rho_0$$

Compare acceleration of a sphere
of the same mass with the real acceleration

$$g_{x,\,sphere} = \frac{GM}{a^2} \iff g_x = 2\pi G\rho_0 A a$$

For $e \simeq 1 \Rightarrow A \simeq \frac{\pi}{2}\sqrt{1-e^2} = \frac{\pi}{2}\cdot\frac{c}{a} \Rightarrow g_x = \pi^2 G\rho_0 c \Rightarrow$

$$g_x = \frac{3}{4}\pi \frac{GM}{a^2} \qquad \frac{3}{4}\pi \simeq 2.356$$

Thus, the largest error is $g_{x,\,sphere}/g_x \simeq \frac{1}{2.356}$

The acceleration at point

$x = y = 0\ ,\ z = c$ $\qquad g_z = 4\pi G\rho_0 c$

This is almost the same acceleration as at point

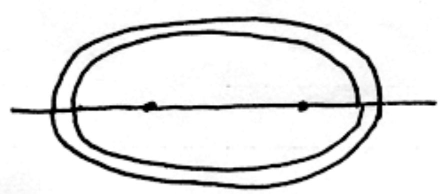
$x = a,\ z = 0$

I — Spheroids with inhomogeneous density distribution

Logic: start with a thin shell (homoeoid) $\Rightarrow$ Find the potential $\Rightarrow$ integrate over all shells $\Rightarrow$ get potential

This is the same logic we had for spherical systems, but now the shell is not a spherical sha

Homoeoid : constant density between two similar spheroids:



$$\frac{R^2}{a^2} + \frac{z^2}{b^2} = 1 \quad \text{and} \quad = (1+\delta)^2$$

$\Rightarrow$ Exterior isopotential surfaces of a homoeoid are spheroids, that are confocal with the shell.

$\Rightarrow$ Inside the shell the potential is constant

Spheroids with isodensity surfaces: (similar spheroids)

$$m^2 = R^2 + \frac{z^2}{1-e^2}$$

$$\Rightarrow \Phi = \Phi(m, e)$$

volume $V = \frac{4}{3}\pi a^2 b$

Interesting application: $\rho(m^2) = \rho_0 \left[1 + \left(\frac{m}{a_0}\right)^2 \right]^{-3/2}$

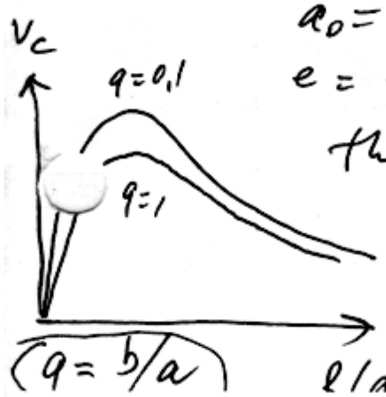
a_0 = core radius $\quad v_c^2(R) \equiv R \frac{\partial \Phi}{\partial R}$

e = eccentricity

this gives $\quad v_c^2(R) = 4\pi G \rho_0 a_0^3 \frac{\sqrt{1-e^2}}{R} k \left[F(\theta_m, k) - E(\theta_m, k) \right]$

incomplete elliptical integrals

$$k \equiv \left[\left(\frac{a_0 e}{R}\right)^2 + 1 \right]^{-1/2}$$

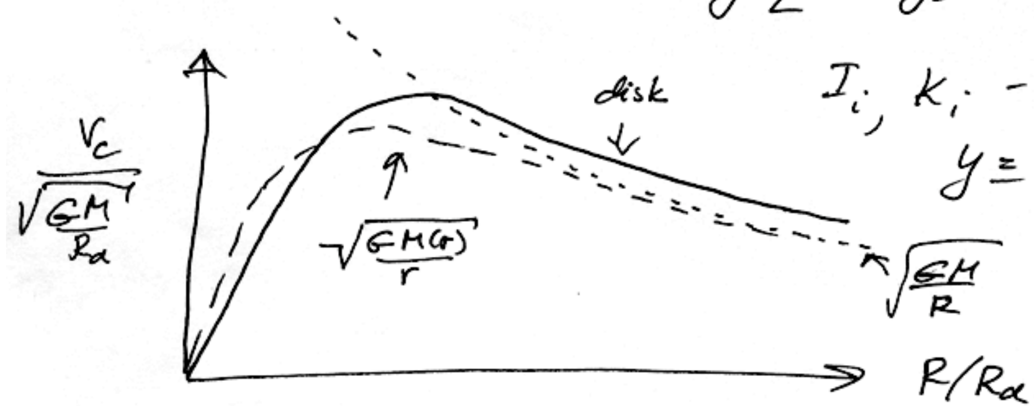


Another example: exponential thin disk

again, the final expression is not easy.

For a thin disk with surface density $\Sigma(R) = \Sigma_0 e^{-R/R_d}$

$$v_c^2(R) = 4\pi G \Sigma_0 R_d \cdot y^2 \left[I_0(y) K_0(y) - I_1(y) K_1(y) \right]$$



I_i, K_i - Bessel functions

$y = \frac{R}{2R_d}$

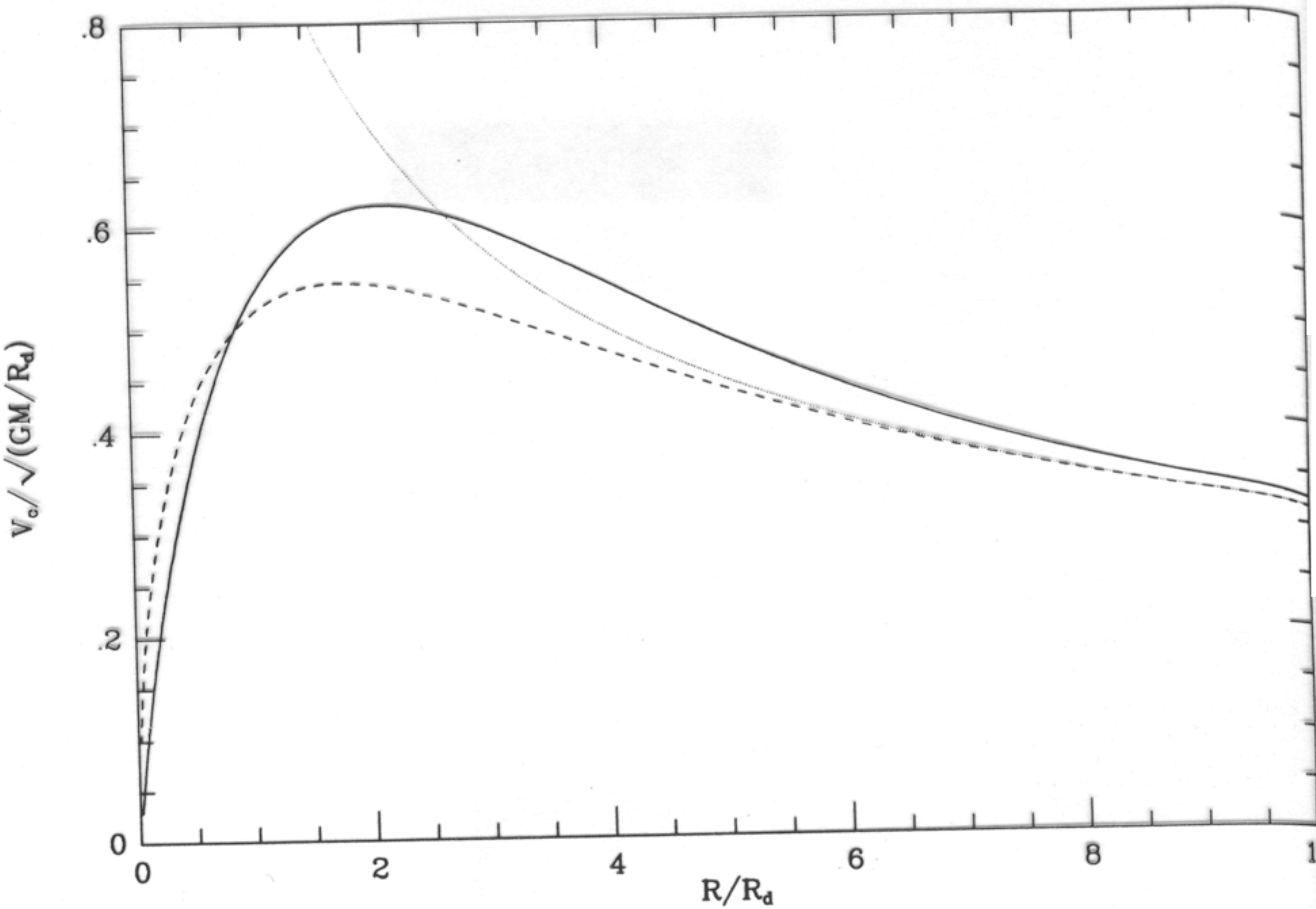


Figure 2-17. The circular-speed curves of: an exponential disk (full curve); a point with the same total mass (dotted curve); the spherical body for which $M(r)$ is given by equation (2-170) (dashed curve).

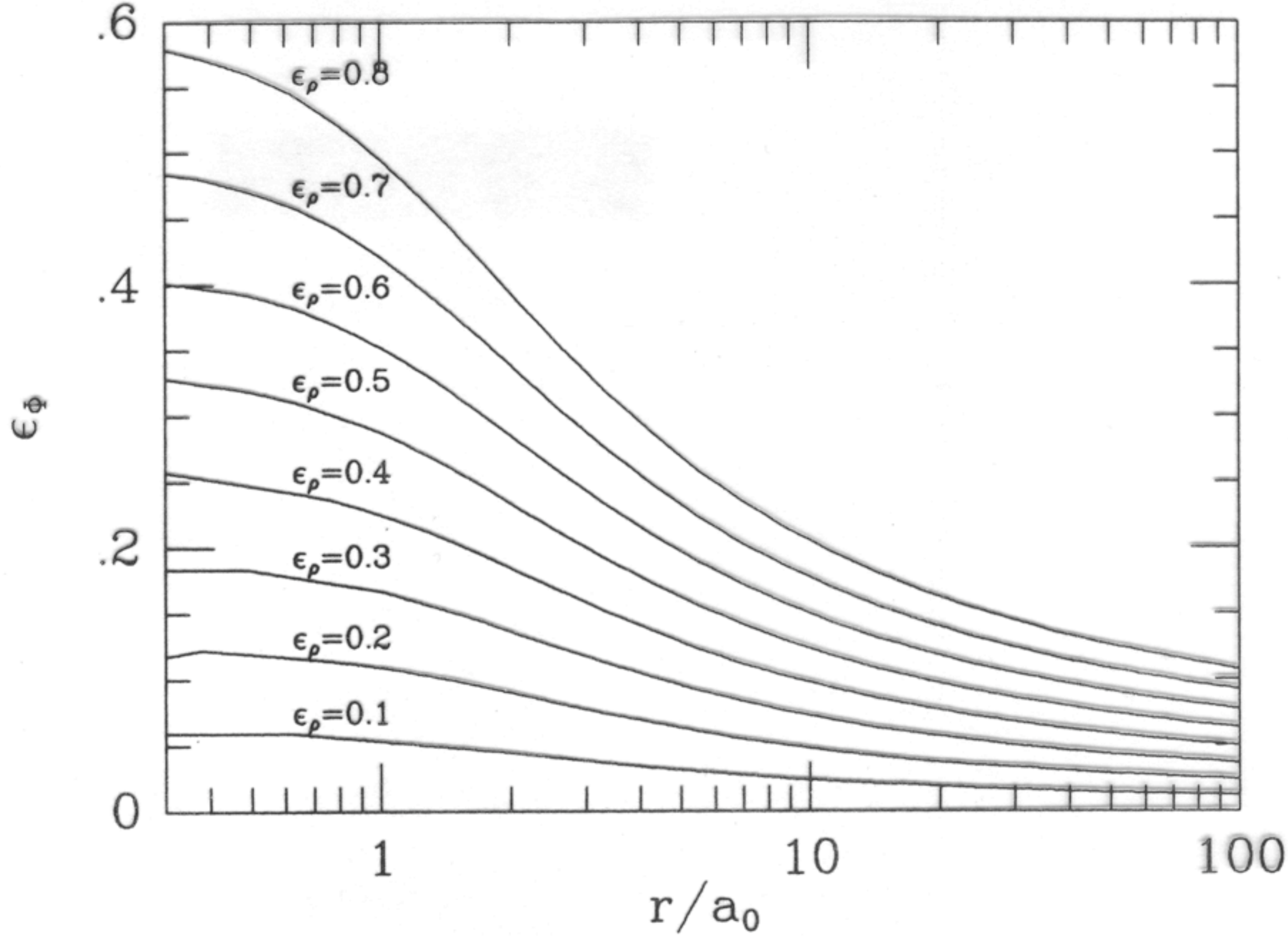


Figure 2-13. The ellipticity ϵ_Φ of an equipotential surface versus the surface's semi-major axis length r. Each curve is labeled by the ellipticity $\epsilon_\rho = 1 - q$ of the body with density (2-92) that generates the corresponding potential. Notice the rapidity with which the equipotential surfaces become spherical at large r/a_0.

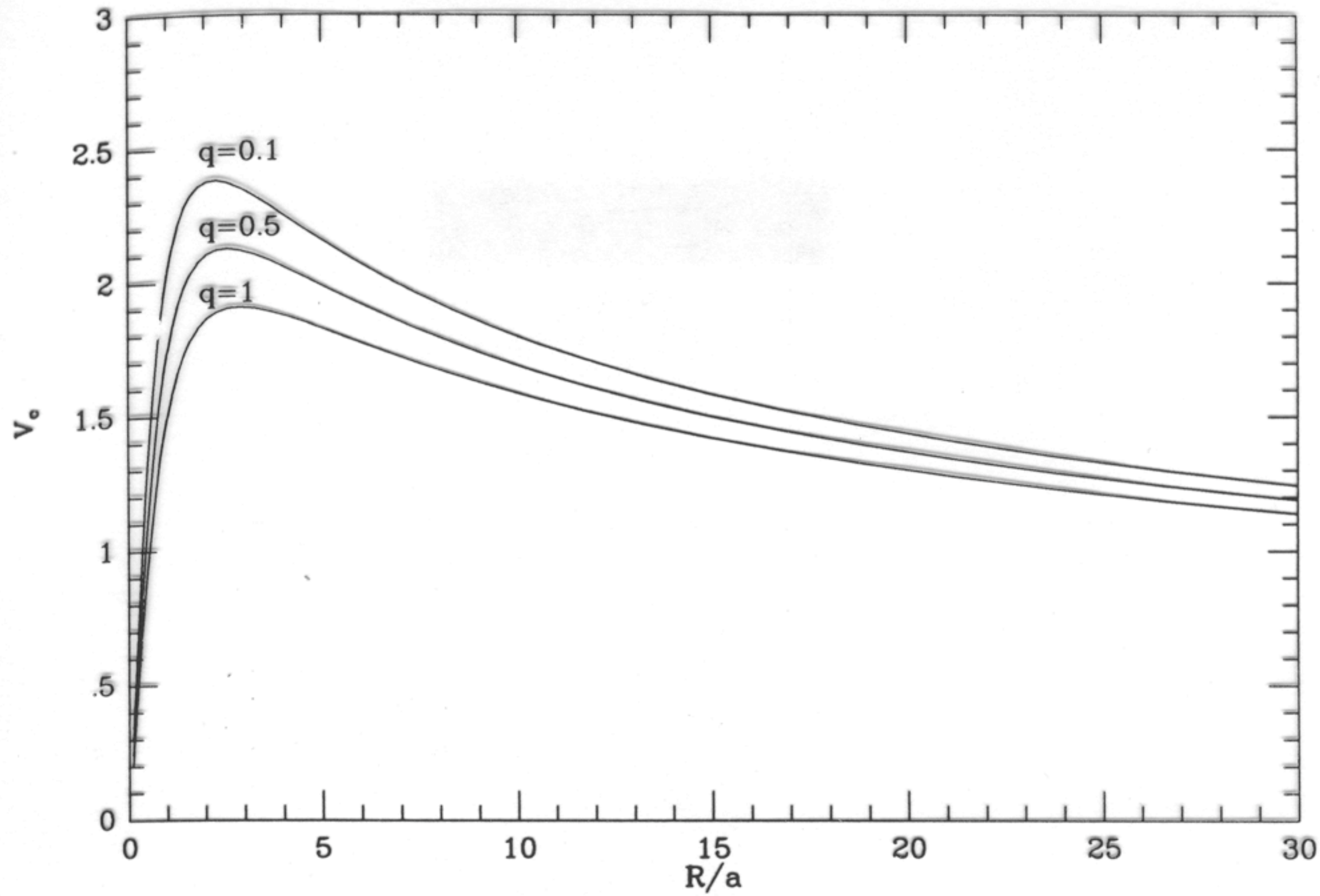


Figure 2-12. Circular speed versus radius for three bodies with the same face-on projected density profile (the modified Hubble profile) but different axis ratios $q = b/a$. Though all three bodies have the same total mass inside a spheroid of given semi-major axis, v_c increases with flattening $1 - q$.