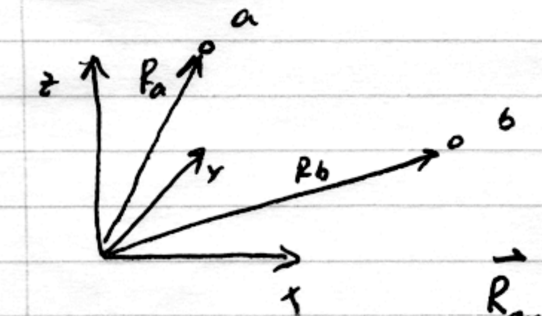
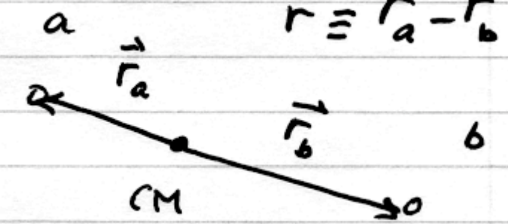
Kepler problem

Kepler problem: motion of two massive objects

Plan:
- coordinates
- $r(\varphi)$
- $r(t)$, $\varphi(t)$
- Kepler's laws, Kepler equation

Coordinates





$$\vec{r} \equiv \vec{r}_a - \vec{r}_b$$

$$\vec{R}_{cm} = \frac{\sum m_i \vec{R}_i}{\sum m_i} = \frac{m_a \vec{R}_a + m_b \vec{R}_b}{m_a + m_b}$$

$$\vec{r}_a = \vec{R}_a - \vec{R}_{cm} = \frac{m_b}{m_a+m_b}(\vec{R}_a - \vec{R}_b) = \frac{m_b}{m_a+m_b}\cdot\vec{r}$$

$$\vec{r}_b = \vec{R}_b - \vec{R}_{cm} = \frac{m_a}{m_a+m_b}(\vec{R}_b - \vec{R}_a) = -\frac{m_a}{m_a+m_b}\cdot\vec{r}$$

Newton's law of Gravity define $\ddot{\vec{r}}_a$, $\ddot{\vec{r}}_b$:

$$(1) \quad \ddot{\vec{r}}_a = -G m_b \frac{\vec{r}}{r^3} = -G(m_a+m_b)\frac{\vec{r}_a}{r^3}$$

$$(2) \quad \ddot{\vec{r}}_b = +G m_a \frac{\vec{r}}{r^3} = -G(m_a+m_b)\frac{\vec{r}_b}{r^3}$$

we subtract eq (2) from eq (1):

$$\ddot{\vec{r}}_a - \ddot{\vec{r}}_b \equiv \ddot{\vec{r}} = -G(m_a+m_b)\frac{\vec{r}}{r^3} \Leftarrow \text{motion in a central force}$$

$$\ddot{\vec{r}} = -\frac{GM}{r^2}\frac{\vec{r}}{r}, \text{ where } M = m_a + m_b$$

<u>Energy</u>: $$\frac{m_a \dot{r}_a^2}{2} + \frac{m_b \dot{r}_b^2}{2} - G\frac{m_a m_b}{r} = E = \text{const}$$

we need to rewrite it in such a way, that only r, not r_a or r_b, is present:

$$\frac{1}{2} m_a \cdot \frac{m_b^2 \dot{r}^2}{(m_a+m_b)^2} + \frac{1}{2} m_b \frac{m_a^2 \dot{r}^2}{(m_a+m_b)^2} - G\frac{m_a m_b}{r} =$$

$$= \frac{m_a m_b}{m_a+m_b} \cdot \frac{\dot{r}^2}{2} - G\frac{m_a m_b}{r} = E \qquad (*)$$

$\mu \equiv \frac{m_a m_b}{m_a+m_b}$ is called reduced mass

eq(*) tells us that Kepler motion is motion of a point with reduced mass μ in Newtonian potential $\sim 1/r$

If we devide eq(*) by μ, we get to the form, convinient for the motion in a central force - energy is in units per unit mass:

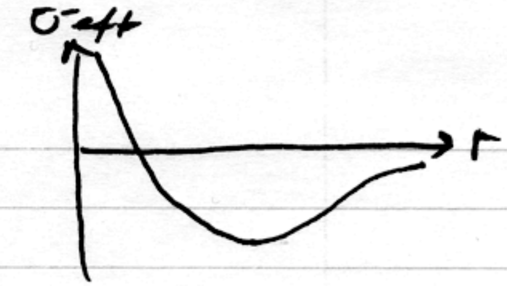
$$\boxed{\frac{\dot{r}^2}{2} - \frac{GM}{r} = \frac{E}{\mu}}$$; we will use E instead of E/μ in following equations!

angular momentum per unit mass:

$$\vec{L} = \vec{r} \times \vec{v}$$

In the plane of the trajectory the radial motion is defined by the effective potential:

$$U_{eff} = -\frac{GM}{r} + \frac{L^2}{2r^2}$$



<u>Trajectory</u> $r(\varphi)$. The general expression for $\varphi(r)$ is

$$\varphi = \int \frac{L}{r^2} \frac{dr}{\sqrt{2(E-U_{eff})}} + \text{const} =$$

$$= \int \frac{L}{r^2} \frac{dr}{\sqrt{2E + \frac{2GM}{r} - \frac{L^2}{r^2}}} + \text{const}$$

the integral is $\arccos\left[\frac{\alpha}{r} + \beta\right]$, where

$$\alpha = -\beta \frac{L^2}{GM}; \quad \beta = -\frac{1}{\sqrt{1+\frac{2EL^2}{(GM)^2}}}$$

Thus,

(*) $$\varphi = \arccos\left\{\frac{1}{e}\left[\frac{L^2}{GMr} - 1\right]\right\},$$ where

$$e \equiv \sqrt{1 + \frac{2EL^2}{(GM)^2}}$$

If we introduce $p \equiv \frac{L^2}{GM}$, eq (*) can be inverted:

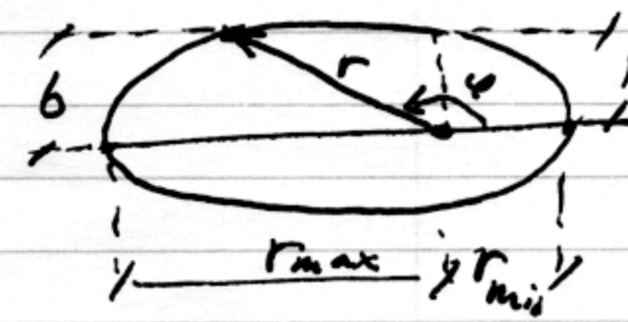
$$\boxed{\frac{p}{r} = 1 + e\cos\varphi}$$

Solution in the form (*) is only valid for the case when e is real, which is always true.

If $E<0$, $\quad e \equiv \sqrt{1+\frac{2EL^2}{(GM)^2}} < 1$

In this case the trajectory is an ellipse. When $\varphi=0$, $r=r_{min}$. When $\varphi=\pi$, $r=r_{max}$

$$r_{min}=\frac{p}{1+e}, \qquad r_{max}=\frac{p}{1-e}$$



Large semi-axis a is

$$a=\frac{r_{min}+r_{max}}{2}=\frac{p}{1-e^2}$$

Thus,

$$r_{min}=a(1-e), \qquad r_{max}=a(1+e)$$

$$\boxed{a=-\frac{GM}{2E}}$$

$$p=\frac{L^2}{GM}, \quad e^2=1+\frac{2EL^2}{(GM)^2}$$

$$\boxed{b=\frac{p}{\sqrt{1-e^2}}=\frac{L}{\sqrt{-2E}}}$$

Large semiaxis does not depend on angular momentum. It depends only on the total energy E

Trajectory: $r(t)$

We need to use another equation of the motion in a central force

$$\frac{dr}{dt} = \sqrt{2(E - U_{eff})} = \sqrt{2E + \frac{2GM}{r} - \frac{L^2}{r^2}}$$

integrating this equation, we get

$$t = \int \frac{dr}{\sqrt{-2E}\sqrt{-1 + \frac{L^2}{2Er^2} - \frac{GM}{E\cdot r}}} = \frac{1}{\sqrt{-2E}} \int \frac{r\,dr}{\sqrt{\frac{L^2}{2E} - \frac{GM}{E} r - r^2}}$$

Let's change our notations from E and L to a and e:

$$\frac{L^2}{2E} - \frac{GM}{E} r - r^2 = \underbrace{-r^2 + 2ar - a^2}\underbrace{+ a^2 + \frac{L^2}{2E}} = -(r-a)^2 + a^2e^2$$

Thus, $$t = \frac{1}{\sqrt{\frac{GM}{a}}} \int \frac{r\,dr}{\sqrt{a^2e^2 - (r-a)^2}}$$

we now change variables: $r - a \equiv -ae\cos\xi$

$dr = ae\sin\xi$

$\xi = 0 - 2\pi$ - eccentric anomaly

$\sqrt{a^2e^2 - (r-a)^2} = ae\sin\xi$

The integral can be taken:

$$t = \frac{a}{\sqrt{\frac{GM}{a}}} \cdot \int (1 - e\cos\xi)\,d\xi = \frac{1}{\sqrt{\frac{GM}{a^3}}}(\xi - e\sin\xi)$$

Trajectory is found in parametric form:

$$\boxed{r = a(1 - e\cos\xi), \text{ where } t = \frac{1}{\sqrt{\frac{GM}{a^3}}}(\xi - e\sin\xi)}$$

Third Kepler's law: $\xi = 2\pi$

$$t \equiv P = \frac{1}{\sqrt{\frac{GM}{a^3}}} \cdot 2\pi$$

This can be rewritten in a more conventional form:

$$\| G(M_a + M_b) P^2 = (2\pi)^2 a^3 \|$$

Kepler equation

The second of equations for trajectory

$$(*) \qquad t = \frac{1}{\sqrt{\frac{GM}{a^3}}} \cdot (\xi - e \sin \xi)$$

Can be re-written in different way.

We define

$$n \equiv \sqrt{\frac{GM}{a^3}} \text{ - mean motion (or mean frequency)}$$

$$\mathbb{M} \equiv nt \text{ - mean anomaly}$$

Then (eq*) takes form of Kepler equation:

$$\boxed{\xi - e \sin \xi = \mathbb{M}}$$

Solution of this equation is the following series:

$$\xi = \mathbb{M} + \sum_{k=1}^{\infty} \frac{2}{k} J_k(ke) \sin k\mathbb{M}, \text{ where}$$

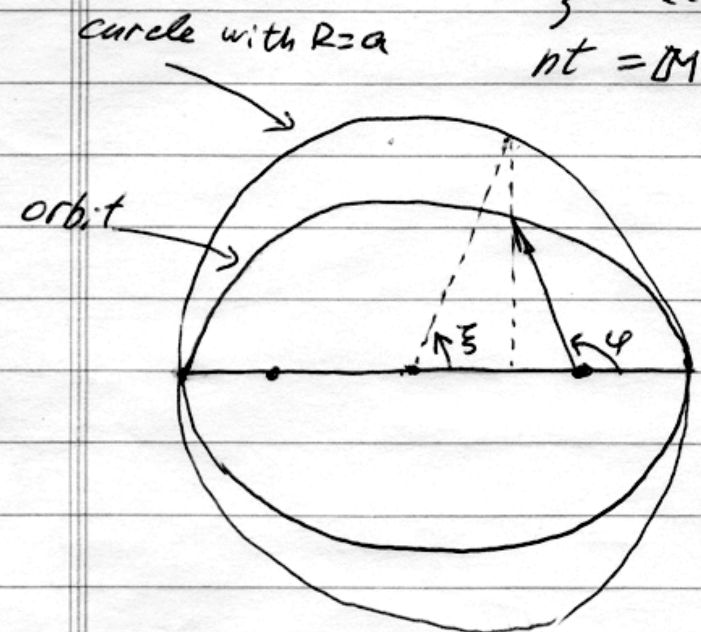
$J_e(x)$ - Bessel functions

Various Anomalies

φ – true anomaly

ξ – eccentric anomaly

$nt = M$ – mean anomaly



Orbital elements

if at some moment of time we have $\vec{r}$ and $\dot{\vec{r}}$, find elements of the orbit a, e

$$\vec{L} = \vec{r} \times \dot{\vec{r}} = \text{angular momentum}$$

$$E = \frac{\dot{\vec{r}}^2}{2} + U(r) \, ; \quad U(r) = -\frac{GM}{r}$$

semimajor axis is defined by $E = -\dfrac{GM}{2a}$, which gives

$$\left\| \frac{1}{a} = \frac{2}{r} - \frac{|\dot{\vec{r}}|^2}{GM} \right\|$$

now, eccentricity is

$$\boxed{e^2 = 1 + \frac{2EL^2}{(GM)^2}}$$