Equations of hydrodynamics: eulerian and lagrangian approaches

Hydrodynamics

=> What are the conditions for using equations of hydrodynamics?

- scales are much larger than the mean-free-path of particles in the medium
- time-scales are longer than the time-scale of relaxation
- volumes are larger than some 'elementary volume': volume is large enough to have many particles inside it and, at the same time, small enough to have small gradients across the 'elementary volume'

Thermal velocities of particles: $v_{thermal} = \left(\frac{3kT}{m}\right)^{1/2}$

For a neutral gas the mean-free-path $l = \frac{1}{n\sigma}$, where σ is the cross-section of atoms.

For air: $\sigma = 4.4 \cdot 10^{-16}\ cm^2$, $n \simeq 10^{19}\ cm^{-3}$

For plasma: two-body relaxation time for electron-proton collisions is

$$t_{e-p} = \frac{3}{4}\left(\frac{m_e}{2\pi}\right)^{1/2}\frac{(kT)^{3/2}}{e^4 n_e \ln\Lambda} \simeq 0.275\ \frac{T^{3/2}}{n_e \ln\Lambda}$$

Here $m_e = \frac{m_H}{1836}$ = mass of an electron, $e = 4.8 \cdot 10^{-10}$ = electron charge

n_e = number density of electrons

$\ln\Lambda$ = Coulomb logarithm ($\simeq 10-20$)

System	parameters	v_{th}	mean-free path
air, room Temp.	$n = 10^{19}\,cm^{-3}$, $T = 300K$ $\sigma = 4.4\,10^{-16}\,cm^2$	$5\cdot 10^4\,cm/s = 0.5\,km/s$ $m = 29\,m_H$	$2\cdot 10^{-4}\,cm$
ISM, disk of Galaxy	$n_e = 1\,cm^{-3}$ m_e, $T = 10^4\,K$	$7\cdot 10^7\,cm/s = 700\,km/s$	$10^{12}\,cm = 3\cdot 10^{-7}\,pc$ $t_{e-p} = 10^4\,sec$
Clusters, gas	$n_e = 10^{-3}$ $T = 10^8\,K$ electrons	$7\cdot 10^9\,cm = 0.25\,c$	$10^{23}\,cm = 30\,kpc$ $t_{e-p} = 1.4\cdot 10^{13}\,s =$ $= 4\cdot 10^5\,yrs$

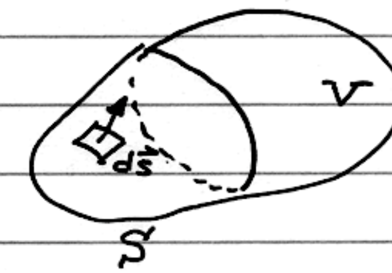
Equations of hydrodynamics:

Some math. relations

$\vec{a}$ is a vector field ; b is a scalar field.

If Volume V is surrounded by closed surface S, then

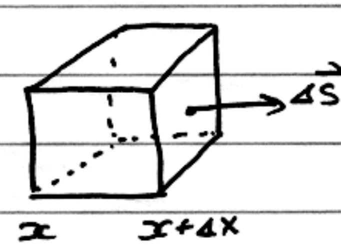
$$\Rightarrow \oint_S \vec{a}\cdot d\vec{s} = \int_V \operatorname{div}\vec{a}\cdot dV$$



$$\Rightarrow \oint_S b\, d\vec{s} = \int_V \operatorname{grad} b \cdot dV$$

$$\Rightarrow \frac{d..}{dt} = \frac{\partial ..}{\partial t} + (\vec{v}\vec{\nabla})..$$

Take small cube.



$$\vec{a}(x+\Delta x, y, z) \simeq \vec{a}(x,y,z) + \frac{\partial \vec{a}}{\partial x}\Delta x$$

$$\oint_{\text{cube}} \vec{a}\, d\vec{s} = \sum_{\substack{\text{all}\\ \text{surfaces}}} \left(a_x \Delta S_x + a_y \Delta S_y + a_z \Delta S_z\right) =$$

$$= a_x(x+\Delta x, y, z)\Delta S_x - a_x(x,y,z)\Delta S_x +$$

$$+ a_y(x, y+\Delta y, z)\Delta S_y - a_y(x,y,z)\Delta S_y +$$

$$+ a_z(x, y, z+\Delta z)\Delta S_z - a_z(x,y,z)\Delta S_z =$$

$$= \frac{\partial a_x}{\partial x}\Delta x \Delta S_x + \frac{\partial a_y}{\partial y}\Delta y \Delta S_y + \frac{\partial a_z}{\partial z}\Delta z \Delta S_z = \operatorname{div}\vec{a}\ \Delta V$$

approximate volume V by many small boxes with ΔV each.
Count contributions of small boxes and set $\Delta V \to 0$

$$\Rightarrow \oint b\, d\vec{s} = \sum_{\substack{\text{surfaces}\\ \text{small box}}} b\Delta\vec{s} = \frac{\partial b}{\partial x}\Delta x \Delta S_x \vec{i} + \frac{\partial b}{\partial y}\Delta y \Delta S_y \vec{j} + \frac{\partial b}{\partial z}\Delta z \Delta S_z \vec{k} =$$

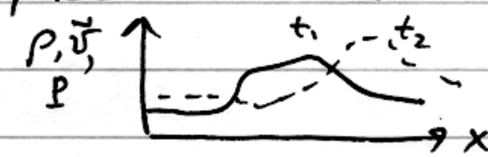
$$= \operatorname{grad} b \cdot \Delta V$$

$\Rightarrow$ Fluid element changes its position by $\Delta\vec{r}$ in Δt. Any property f of the element also changes.

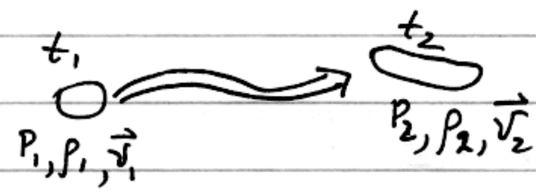
$$\Delta f = \frac{\partial f}{\partial t}\Delta t + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z \Rightarrow \frac{df}{dt} = \frac{\partial f}{\partial t} + (\vec{v}\vec{\nabla})f$$

Equations of hydrodynamics: – Euler approach
– Lagrange approach

Euler: Find evolution of parameters of fluid at each point in space and time



Lagrange: Find evolution of each fluid element:



Euler formalism: equations of hydrodynamics

Continuity equation. Take volume element V. Mass inside V is

$$\int \rho \, dV$$

Through a surface element $d\vec{s}$ of the volume flows out mass $\rho \vec{v} d\vec{s} dt$. Total mass leaving the volume V in dt is

$$\oint \rho \vec{v} d\vec{s} dt$$

The total rate of change of mass is

$$\frac{\partial}{\partial t}\left[\int \rho \, dV\right] = -\oint \rho \vec{v} d\vec{s} \qquad \text{Note the sign}$$

Change the integral on the right from surface to volume and change the order of integration and time derivative

$$\int \left(\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \vec{v}) \right) dV = 0$$

This relation must be valid for any volume V. This can be true only if the integrand is equal to <u>zero</u>

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla(\rho \vec{v}) = 0}$$

Rewrite this equation: $\frac{\partial \rho}{\partial t} + \vec{v}\nabla\rho + \rho\nabla\vec{v} = 0$

The first two terms are equal to $\frac{\partial \rho}{\partial t} + \vec{v}\nabla\rho = \frac{d\rho}{dt}$

Thus,

$$\frac{d\rho}{dt} + \rho\nabla v = 0$$

For incompressible fluid (e.g. water) $\frac{d\rho}{dt} = 0$. In this case $\operatorname{div}\vec{v} = 0$.

This tells us that $\operatorname{div}\vec{v}$ is a measure of compressibility of fluid.

<u>Euler equation</u>. Take volume V. Pressure force acting on the volume from fluid, which surrounds the volume is

(*) $$-\oint p\, d\vec{s}$$ Note the sign

$\oint_S p\, d\vec{s} = \int_V \operatorname{grad} p\, dV$. The rate of change of the momentum of fluid inside V is

(**) $\int \rho \frac{d\vec{v}}{dt} \cdot dV$. This gives $\rho\frac{d\vec{v}}{dt} = -\operatorname{grad} P$

Change dt to ∂t:

$$\boxed{\frac{\partial \vec{v}}{\partial t} + (\vec{v}\nabla)\vec{v} = -\frac{1}{\rho}\nabla P - \nabla\Phi}$$

where Φ is the grav. potential

Energy equation

We start with the first law of thermodynamics:

$$\frac{d\varepsilon}{dt} + p\frac{dV}{dt} = \frac{dQ}{dt}$$

Here ε is internal energy per unit mass. $V = 1/\rho =$ = specific volume. dQ is external energy per unit mass. Using the continuity and Euler equations, the energy equation can be written in the form

$$(*) \qquad \frac{\partial}{\partial t}\left(\rho\varepsilon + \rho\frac{v^2}{2}\right) + \nabla\left(\rho\vec{v}\left(\varepsilon + \frac{v^2}{2}\right) + p\vec{v}\right) = \rho\frac{dQ}{dt}$$

We note that $\left(\varepsilon + \frac{v^2}{2}\right)$ is the total energy = thermal + kinetic

$$E = \rho\left(\frac{v^2}{2} + \varepsilon\right) = \text{total energy per unit volume}$$

$$(**) \qquad \frac{\partial E}{\partial t} + \operatorname{div}(E\vec{v}) = -\operatorname{div}(p\vec{v}) + \rho\frac{dQ}{dt}$$

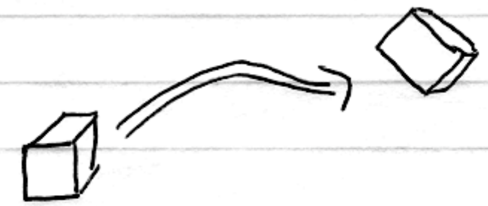
For an adiabatic process we can use the equation of entropy conservation: $\frac{ds}{dt} = 0$,

where s is the specific entropy. This can be written in the form

$$\frac{\partial s}{\partial t} + \vec{v}\,\operatorname{grad} s = 0$$

Lagrangian approach to hydrodynamics

=> follow motion of a fluid element. Mass of the element is preserved, but its position in space and its volume change as the element moves

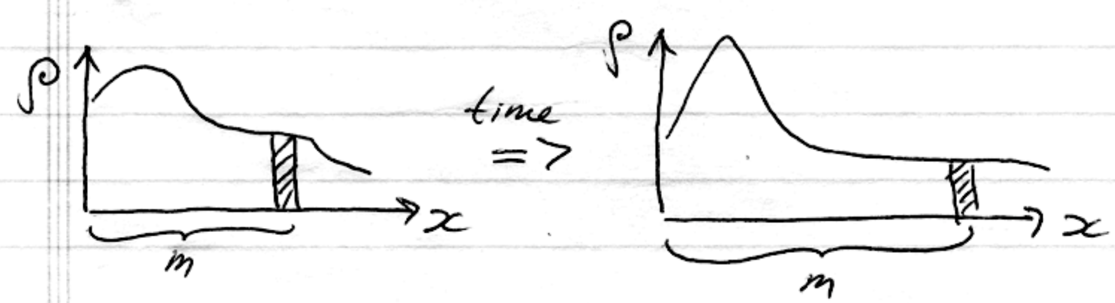


We must introduce a way to identify the element and to distinguish it from other elements. Let $\vec{q}$ be a label ("coordinate"), which characterizes the element. There are different ways of chosing $\vec{q}$. For example, $\vec{q}$ can be position of the element at some initial moment of time.

Motion of the fluid is described by the following set of relations:

$$\rho = \rho(t, \vec{q}) \quad = \text{density}$$
$$\vec{v} = \vec{v}(t, \vec{q}) \quad = \text{velocity}$$
$$P = P(t, \vec{q}) \quad = \text{pressure}$$
$$\vec{x} = \vec{x}(t, \vec{q}) \quad = \text{position in space}$$

xamples: (I) Plane motion: there is only one coordinate x



mass (m) between (any) elements is preserved, but positions of the elements changes with time. We can use (m) as lagrangian coordinate q.

(2)

For simplicity we assume that the first element does not move.

$$m(x) = \int_{x_1}^{x} \rho\, dx, \qquad dm = \rho\, dx$$

Mass m is our independent variable.

The continuity equation in Eulirian coordinates is

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{v}) = 0$$

We need to rewrite it so that time derivative is $\frac{d}{dt}$, not $\frac{\partial}{\partial t}$:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{v}) = \frac{\partial \rho}{\partial t} + \vec{v}\operatorname{grad}\rho + \rho \operatorname{div}\vec{v} = \frac{d\rho}{dt} + \rho \operatorname{div}\vec{v} = 0$$

In 1D case this equation takes form:

$$\frac{d\rho}{dt} + \rho \frac{\partial v}{\partial x} = 0$$

Now we need to change variables from x to m:

use $dm = \rho\, dx$. We get continuity equation in Lagrangian coordinates:

$$\boxed{\frac{1}{\rho^2}\frac{d\rho}{dt} = -\frac{\partial v(m,t)}{\partial m}}$$

(3)

The equation of motion in lagrangian coordinates is

$$\frac{dv}{dt} = -\frac{\nabla P}{\rho} - \nabla\varphi$$

Here we need to change $\frac{\partial}{\partial x}$ to $\rho\frac{\partial}{\partial m}$:

$$\frac{1}{\rho}\frac{\partial P}{\partial x} = \frac{\partial P(m,t)}{\partial m} \; ; \; \frac{\partial\varphi}{\partial x} = \rho\frac{\partial\varphi(m,t)}{\partial m}$$

Thus, the final form of the equation of motion in lagrangian coordinates takes the form

$$\boxed{\frac{dv}{dt} = -\frac{\partial P}{\partial m} - \rho\frac{\partial\varphi}{\partial m}}$$

Energy equation

$$\frac{d\varepsilon}{dt} + p\frac{dV}{dt} = \frac{dQ}{dt}$$

For adiabatic process: $\frac{dS}{dt} = 0$

Another choice of coordinates: use initial coordinate x as the lagrangian coordinate. If $\rho_0(x)$ is the density distribution at initial moment, than mass conservation

$$\rho_0 dx = \rho(r,t)dr$$

Equations of hydrodynamics are:

$$\left\| \begin{aligned} &\frac{1}{\rho^2}\frac{d\rho}{dt} = -\frac{\partial v(t,x)}{\partial x}\frac{1}{\rho_0(x)} \\ &\frac{dv(t,x)}{dt} = -\frac{1}{\rho_0}\frac{\partial P}{\partial x} - \frac{\rho(x,t)}{\rho_0(x)}\frac{\partial\varphi}{\partial x} \end{aligned} \right\|$$

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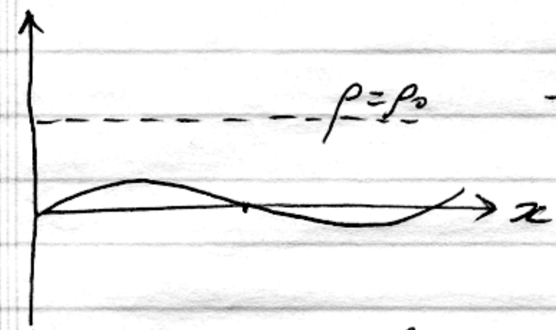
Example: motion of cold gas in one-dimensional case

$$P = 0 , \varphi = 0$$

Initial conditions: $-v = v_0 \sin(kx)$,

v_0 and k are constants



- initial density $\rho = \rho_0 = const$

How the gas flows at later times?

$$\frac{1}{\rho^2}\frac{d\rho}{dt} = -\frac{\partial v}{\rho_0 \partial x} = -\frac{v_0}{\rho_0} k\cos(kx)$$

The r.h.s. does not depend on time t. Thus, we can integrate this equation

$$\frac{1}{\rho_0} - \frac{1}{\rho(x,t)} = -\frac{v_0}{\rho_0} k\cos(kx)(t-t_0)$$

$$\Rightarrow \boxed{\rho = \frac{\rho_0}{1+v_0 k(t-t_0)\cos(kx)}}$$

Find relation between lagrangian coordinate x and eulirian coordinate r: From continuity equation

$$\rho(x,t)dr = \rho_0(x)dx \Rightarrow \frac{dr}{dx} = \frac{\rho_0}{\rho(x,t)}$$

Integrate over x:

$$\boxed{r = x + v_0(t-t_0)\sin(kx)}$$

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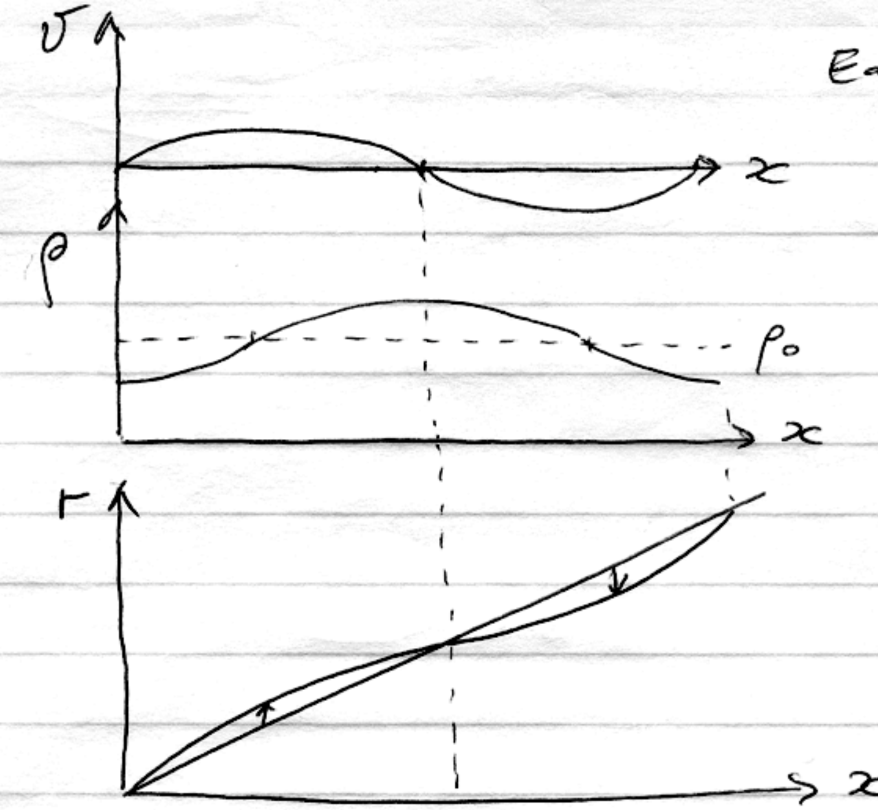
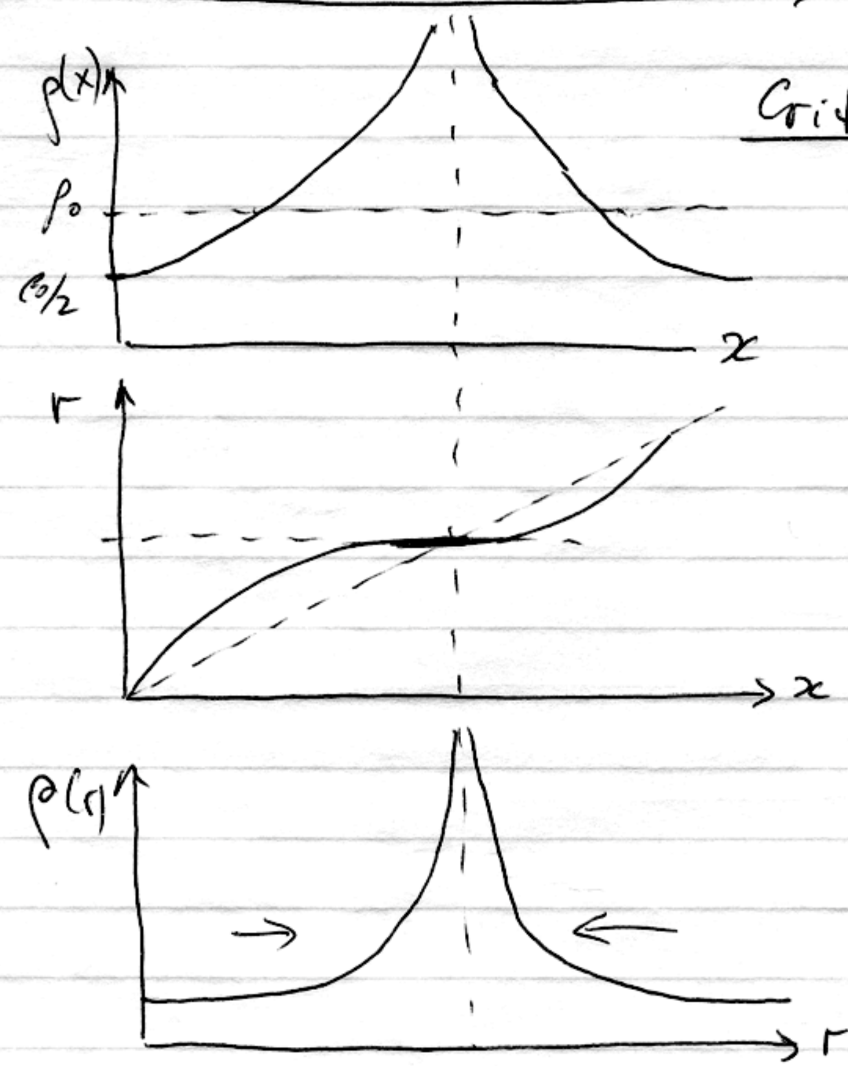


Early moments of time:

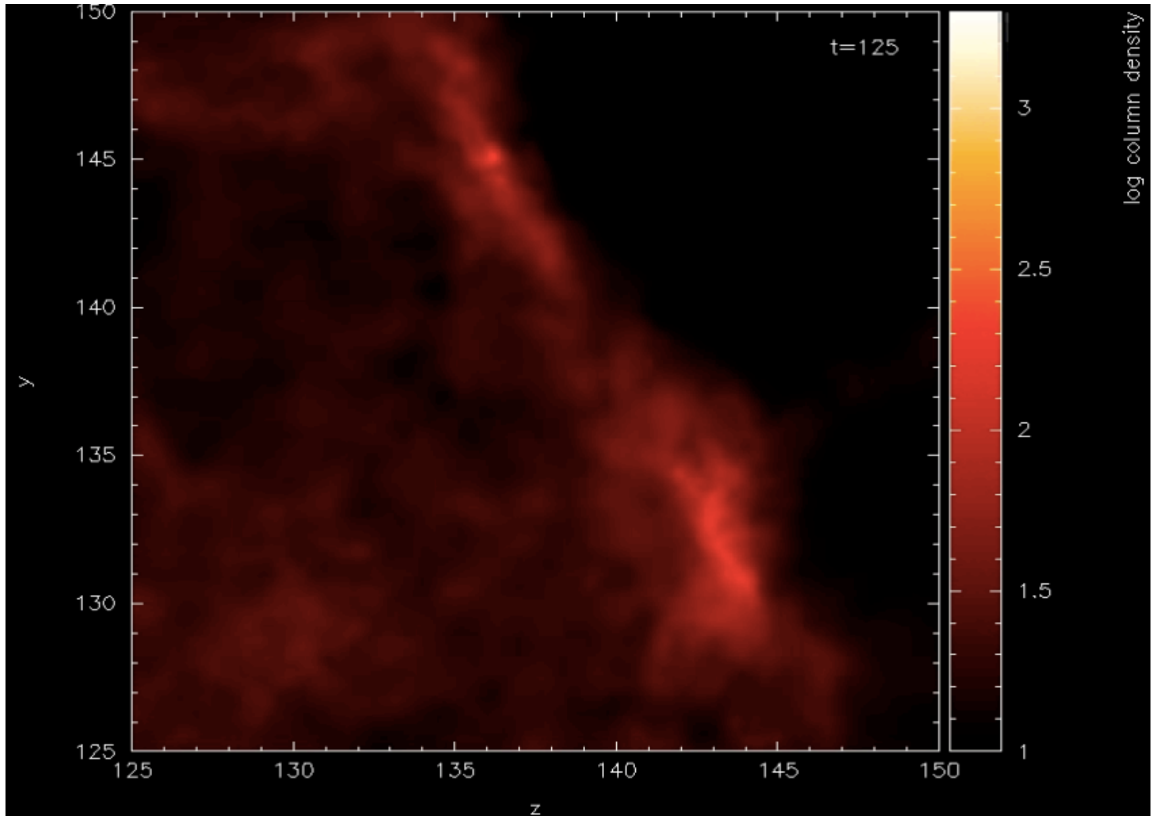
$v_0 K(t-t_0) \ll 1$

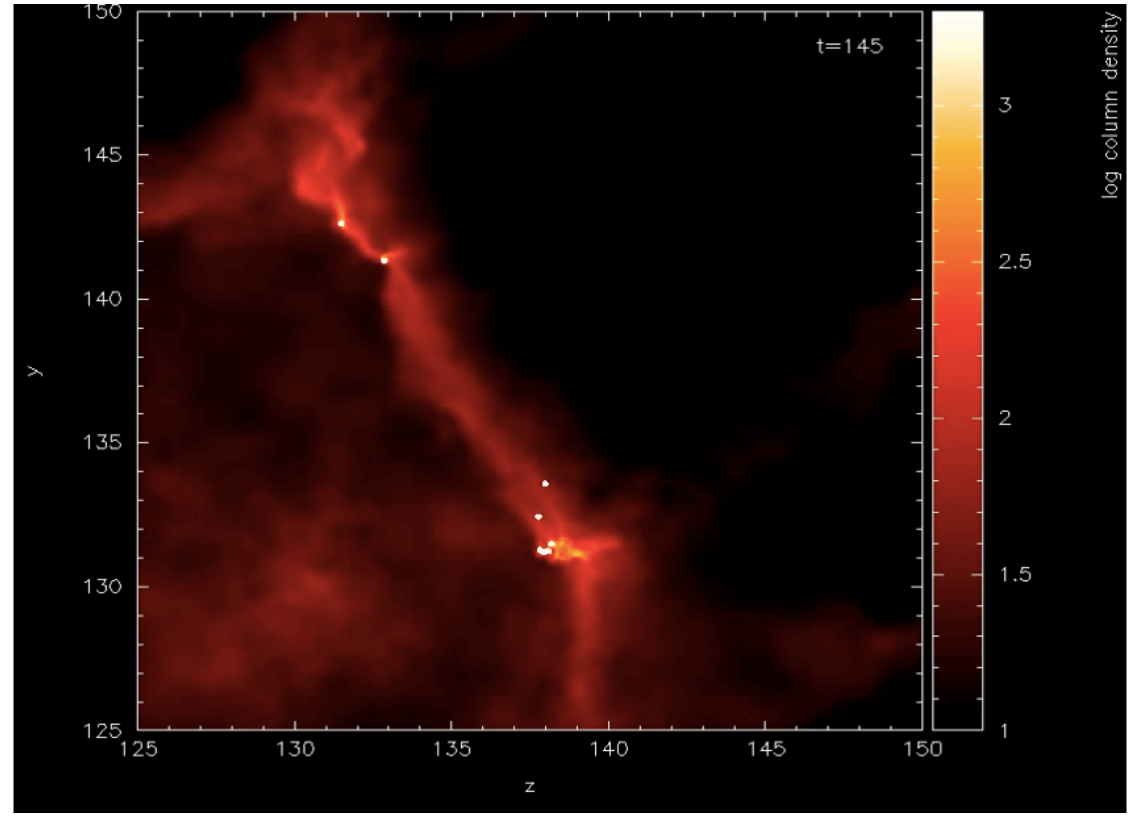


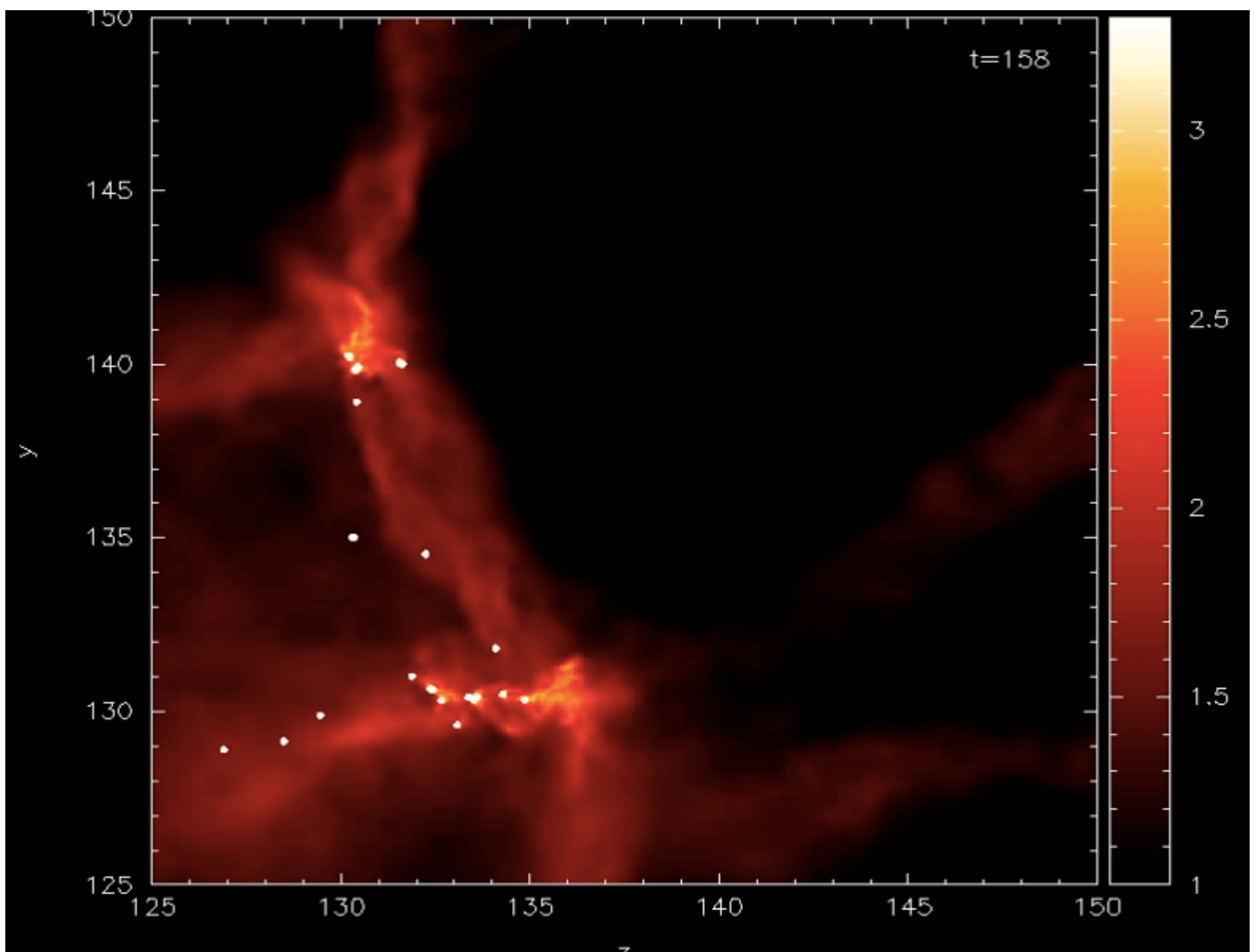
Critical moment:

$v_0 k(t-t_0) = 1$

infinite density at one point







Column density plot of Cloud 1 in the *y*-*z* plane at $t = 19.1$ Myr, integrating over the central 16 pc along the *x*-direction. The dots show the stellar objects (sink particles). The electronic version of this figure shows an animation of this region from $t = 16.6$ to 19.9 Myr. The column density is in code units, which correspond to 9.85×10^{19} cm^{-2}.

Model of star formation in Molecular clouds