Epicycle Approximation

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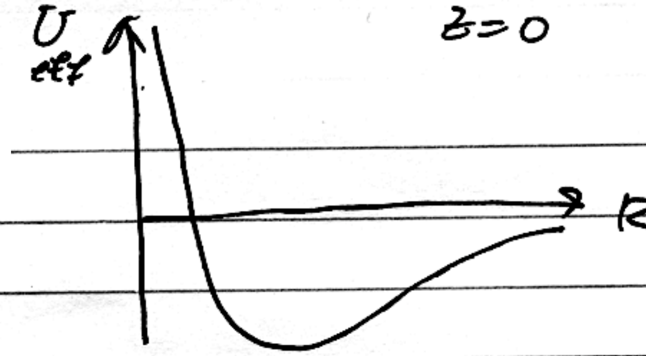
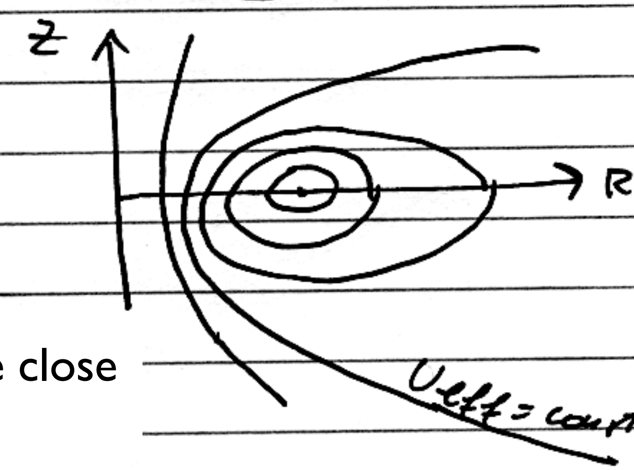


Equations: $\ddot{R} = -\dfrac{\partial U_{eff}}{\partial R}$

$$\ddot{z} = -\frac{\partial U_{eff}}{\partial z}$$

$$U_{eff} = U(R,z) + \frac{L_z^2}{2R^2}$$



In Epicycle Approximation we consider orbits, which are close to the minimum of the effective potential

The fact that U_{eff} is minimum $\Rightarrow$

$$\Rightarrow \frac{\partial U_{eff}}{\partial R} = 0 = \frac{\partial U}{\partial R} - \frac{L_z^2}{R^3}$$

Relation between U and acceleration: $g_R = -\dfrac{\partial U}{\partial R}$

Thus, at the minimum:

$$g_R = \frac{L_z^2}{R^3} = \frac{v_\varphi^2}{R} \;\|$$

This is a circular orbit: $R = R_g = const$

Now, we make a small perturbation (kick it).

A deviation from the circular orbit is

$$x(R,z,t) \equiv R - R_g$$

Expand $U_{eff}(R,z)$ into a Taylor series about $R = R_g$:

$$U_{eff} = \frac{1}{2}\frac{\partial^2 U_{eff}}{\partial R^2}\Big|_{\substack{R_g\\ z=0}} x^2 + \frac{1}{2}\frac{\partial^2 U_{eff}}{\partial z^2} z^2 + \dots$$

Rewrite this as

$$U_{eff} \simeq \frac{1}{2} \kappa^2 x^2 + \frac{1}{2} \nu^2 z^2,$$

where $\kappa \equiv \sqrt{\frac{\partial^2 U_{eff}}{\partial R^2}}$, $\nu \equiv \sqrt{\frac{\partial^2 U_{eff}}{\partial z^2}}$

κ is called an epicycle frequency
ν is vertical frequency

$$\kappa^2 = \frac{\partial^2 U_{eff}}{\partial R^2} = \frac{\partial^2 U}{\partial R^2} + \frac{3 L_z^2}{R_g^4}$$

Introduce $\Omega(R)$ = angular velocity of the circular orbit. For a circular orbit we have:

$$\Omega^2 R = \frac{\partial U}{\partial R} = \frac{L_z^2}{R_g^3} \Rightarrow \frac{\partial^2 U}{\partial R^2} = \Omega^2 + R \frac{d\Omega^2(R)}{dR}$$

and $\frac{L_z^2}{R_g^4} = \Omega^2$

Thus, $$\boxed{\kappa^2 = 4\Omega^2 + R \frac{d\Omega^2}{dR}}$$

For all astronomically interesting cases

$$\Omega < \kappa < 2\Omega$$

Equations of motion for particles close to circular motion are

$$\begin{cases} \ddot{x} = -\varkappa^2 x \\ \ddot{z} = -\nu^2 z \\ \dot{\varphi} = \dfrac{L_z}{R^2} \end{cases}$$

Solution of the equations are

$$\begin{cases} x = B\cos(\varkappa t + C_1) \\ z = A\cos(\nu t + C_2) \\ y = -\dfrac{2\Omega}{\varkappa} B\sin(\varkappa t + C_1) \end{cases}$$