Motion in a central force

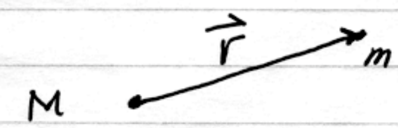
Particle motion in a central force

- acceleration and potential for central force
- angular momentum
- equations for trajectories
- analysis

Definition: Force is central if it depends only radius-vector $\vec{r}$

$$\vec{F} = \Phi(r)\vec{r}$$

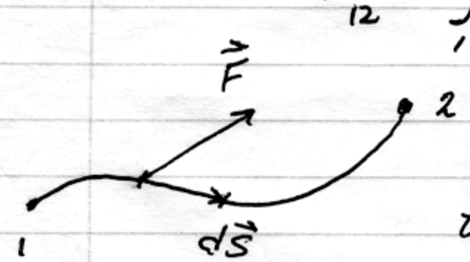
Example: Newtonian force: $\vec{F} = -GMm\dfrac{\vec{r}}{|r|^3}$



Gravitational potential (U or φ):

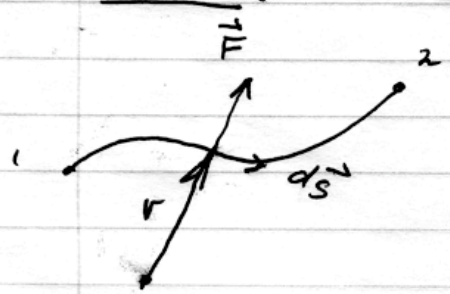
work done by the force on a particle to displace it from position 1 to position 2:

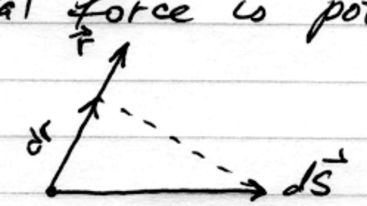
$$W_{12} = \int_1^2 \vec{F}d\vec{s} = \int_1^2 F_x dS_x + F_y dS_y + F_z dS_z$$



If W_{12} does not depend on the path, the force is called conservative or potential. In this case W_{12} is the same for any path joining points 1 and 2

Theorem: a central force is potential





$$\vec{r}d\vec{s} = rdr$$

where dr is the projection of $d\vec{s}$ on $\vec{r}$

By definition $\vec{F} = \Phi(r)\vec{r}$

The work W_{12} can be written as

$$W_{12} = \int_1^2 \vec{F} d\vec{s} = \int_1^2 \Phi(r)\vec{r} \cdot d\vec{s} = \int_{r_1}^{r_2} \Phi(r) r \, dr$$

The last integral depends only on r_1 and r_2, not on a particular path joining 1 and 2. Thus, the force is potential.

We can define gravitational potential as the work done by the force to move a particle of unit mass from $\vec{r}$ to infinity. This is convinient if the system is finite (does not extend to ∞)

$$(*) \qquad U(\vec{r}) = \int_{\vec{r}}^{\infty} \vec{F} d\vec{r} = \int_{\vec{r}}^{\infty} F_x dx + F_y dy + F_z dz$$

If we differentiate $(*)$ in regard to $\vec{r}$, we get inverse relation:

$$(**) \qquad \vec{F} = -\mathrm{grad}\, U(r) \equiv -\vec{\nabla} U(r)$$

In particular case of the central force $(**)$ can be written as

$$(***) \qquad \vec{F} = -\frac{dU(r)}{dr} \frac{\vec{r}}{|r|}$$

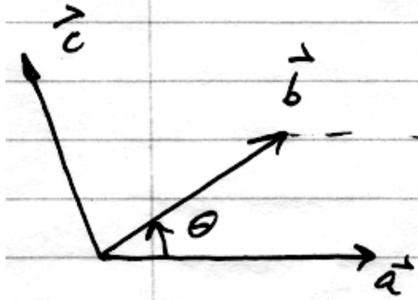
For a particle of unit mass eq$(***)$ means that the trajectory of the particle is

$$(****) \qquad \ddot{\vec{r}} \equiv \frac{d^2\vec{r}}{dt^2} = -\frac{dU}{dr} \frac{\vec{r}}{|\vec{r}|}$$

It is convinient to define $U(r)$ as work per unit mass. Then $(****)$ is valid for any particle.

Specific angular momentum (angular momentum per unit mass) is defined as:

$$\vec{M} = [\vec{r}\,\vec{v}]$$



vector product: $\vec{c} = [\vec{a}\,\vec{b}] \equiv \vec{a} \times \vec{b}$

for $\vec{a} = \{a_x, a_y, a_z\}$, $\vec{b} = \{b_x, b_y, b_z\}$

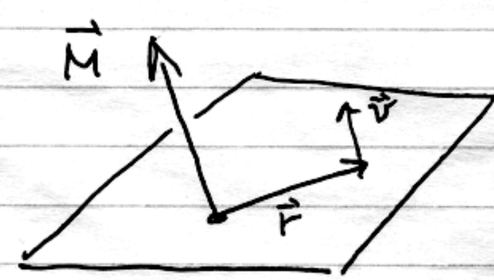
$$\vec{c} = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_x & a_y & a_z \\ b_y & b_y & b_z \end{vmatrix}$$

$|\vec{c}| =$ a b sinΘ= Area of the parallelogram on the plot

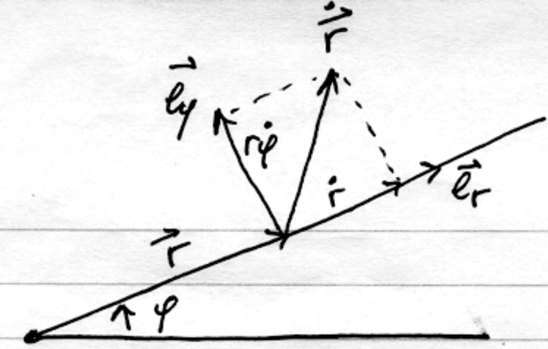
For central force we can find that the derivative of angular momentum with time is:

$$\dot{\vec{M}} = \frac{d\vec{M}}{dt} = \frac{d}{dt}[\vec{r}\,\vec{v}] = \frac{d}{dt}[\vec{r}\,\dot{\vec{r}}] = [\dot{\vec{r}}\,\dot{\vec{r}}] + [\vec{r}\,\ddot{\vec{r}}] = 0$$

Thus, the angular momentum of a particle is conserved. Its trajectory is confined to the same plane: the one, which is orthogonal to vector M



We chose a reference frame, which has $\vec{z}$-component along $\vec{M}$. The trajectory will always stay in the plane xy.
We introduce polar coordinates in the plane.
Let $\vec{e}_r$ and $\vec{e}_\varphi$ be unit vectors along and perpendicular to the radius-vector $\vec{r}$. φ is the polar angle.



Radius-vector of the particle: $\vec r = r\vec e_r$

(*) Velocity of the particle: $\dot{\vec r} = \dot r \vec e_r + r\dot\varphi \vec e_\varphi$

Angular momentum:

$$\vec M = [\vec r\,\dot{\vec r}] = [\vec r, \dot r\vec e_r] + [\vec r, r\dot\varphi\vec e_\varphi] = r^2\dot\varphi[\vec e_r\,\vec e_\varphi]$$

Thus,

$$\vec M = r^2\dot\varphi\,\vec e_z = const$$

and

$$\| r^2\dot\varphi = const \|$$

Equation of motion of a particle in central force is:

(**) $$\ddot{\vec r} = -\vec\nabla U(r)$$

Differentiate eq (*) with time:

$$\ddot{\vec r} = \ddot r\vec e_r + (\dot r\dot\varphi + r\ddot\varphi)\vec e_\varphi + r\dot\varphi\dot{\vec e}_\varphi + \dot r\dot{\vec e}_r$$

From geometry of the problem we find:

$$\dot{\vec e}_r = \dot\varphi\vec e_\varphi, \quad \dot{\vec e}_\varphi = -\dot\varphi\vec e_r$$

Thus, $$\ddot{\vec r} = (\ddot r - r\dot\varphi^2)\vec e_r + (2\dot r\dot\varphi + r\ddot\varphi)\vec e_\varphi$$

For a central force $\vec\nabla U = \frac{dU}{dr}\vec e_r$. Eq (**) now can be written as two equations: one for each component (r or φ)

$$\begin{cases} 2\dot r\dot\varphi + r\ddot\varphi = 0 & (\varphi) \\ \ddot r - r\dot\varphi^2 = -\frac{dU}{dr} & (r) \end{cases}$$

Conservation of angular momentum follows from (φ) equation

$$2\dot{r}\dot{\varphi} + r\ddot{\varphi} = (\dot{\varphi} r^2)^{\cdot} = 0 \Rightarrow \dot{\varphi} r^2 = \text{const}$$

$$\Rightarrow \boxed{\dot{\varphi} = \frac{M}{r^2}}$$

We substitute this equation into the (r) equation and get:

$$\ddot{r} = -\frac{dU}{dr} + r\frac{M^2}{r^4} = -\frac{d}{dr}\left[U + \frac{M^2}{2r^2}\right]$$

Definition:

$$U_{eff} \equiv U(r) + \frac{M^2}{2r^2}$$ = Effective Potential

Now the second equation of motion can be written as:

$$\boxed{\ddot{r} = -\frac{dU_{eff}}{dr}}$$

Equation for radial motion $\qquad \ddot{r} = -\dfrac{dU_{eff}}{dr} \qquad (*)$

can be integrated once. Multiply $(*)$ by $\dot{r}$. On the left hand-side we get $\dot{r}\ddot{r} = \frac{1}{2}(\dot{r}^2)^{\cdot}$.

The right hand-side is $-\dfrac{dU_{eff}}{dr}\dfrac{dr}{dt} = -\dfrac{dU_{eff}}{dt} = -\dot{U}_{eff}$

Note, that this is true only if U_{eff} does not depend on time explicitly. Thus, eq $(*)$ can be written in the form

$$\frac{d}{dt}\left(\frac{\dot{r}^2}{2}\right) = -\frac{dU_{eff}}{dt}$$

This can be integrated:

$$(**) \qquad \frac{\dot{r}^2}{2} + U_{eff} = E,$$ where E is a constant of integration.

The constant E is the total energy per unit mass of the particle:

$$E = \frac{\dot{r}^2}{2} + U(r) + \frac{M^2}{2r^2} = \frac{\dot{r}^2}{2} + \frac{v_\perp^2}{2} + U(r),$$ where

$$v_\perp = \frac{M}{r} = \text{tangential velocity}$$

Trajectory of a particle moving in central force

$$\frac{\dot{r}^2}{2} + U_{eff}(r) = E \Rightarrow \frac{dr}{dt} = \pm\sqrt{2(E - U_{eff}(r))} \quad (*)$$

This can be written in integral form:

$$\int_{t_0}^{t} dt = \pm \int_{r_0}^{r} \frac{dr}{\sqrt{2(E - U_{eff}(r))}}, \text{ where}$$

t_0 and r_0 are initial moment and initial radius

If we choose $t_0 = 0$ when the trajectory has minimum radius,

$$\left\| \; t = \int_{r_0}^{r} \frac{dr}{\sqrt{2(E - U_{eff})}} \; \right\|$$

This equation defines dependence of radius on time. The angle φ can be found from conservation of angular momentum:

$$\frac{d\varphi}{dt} = \frac{M}{r^2} \Rightarrow \overset{\text{from eq}(*)}{dt} = \frac{dr}{\pm\sqrt{2(E - U_{eff})}} \Rightarrow$$

$$\Rightarrow \frac{d\varphi}{dr} = \pm \frac{M}{r^2} \frac{1}{\sqrt{2(E - U_{eff})}}$$

$$\left\| \; \varphi - \varphi_0 = \pm \int_{r_0}^{r} \frac{M}{r^2} \frac{dr}{\sqrt{2(E - U_{eff}(r))}} \; \right\|$$

This is the second equation for the trajectory. It defines φ as a function of radius r

Analysis

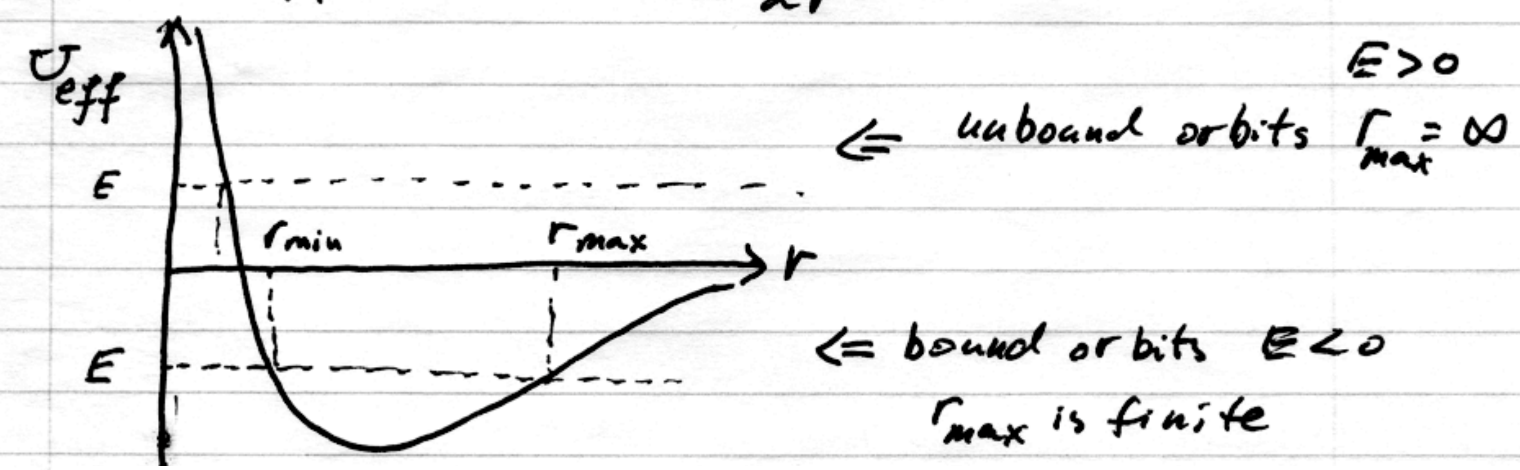
Energy E and angular momentum L are defined by initial conditions

at initial moment $t = t_0$ we have $r_0, \varphi_0, v_{r,0}, v_{\perp,0}$

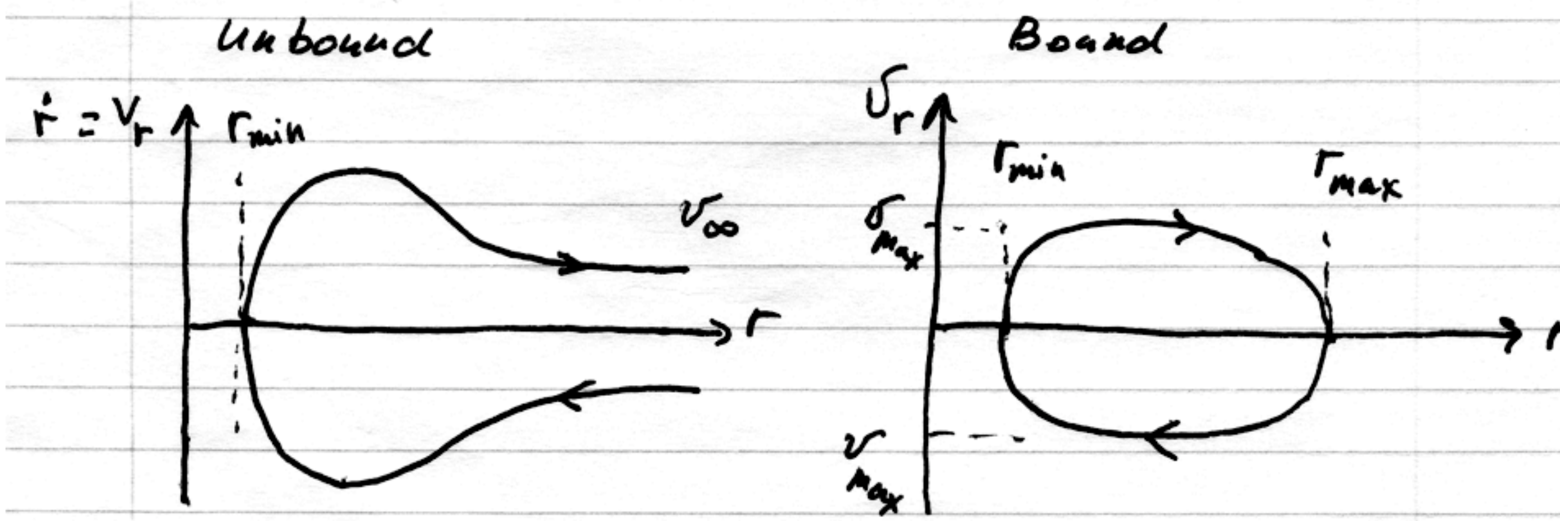
$$E = \frac{v_{r,0}^2}{2} + \frac{v_{\perp,0}^2}{2} + U(r_0) \; ; \; M = v_{\perp,0} \cdot r_0$$

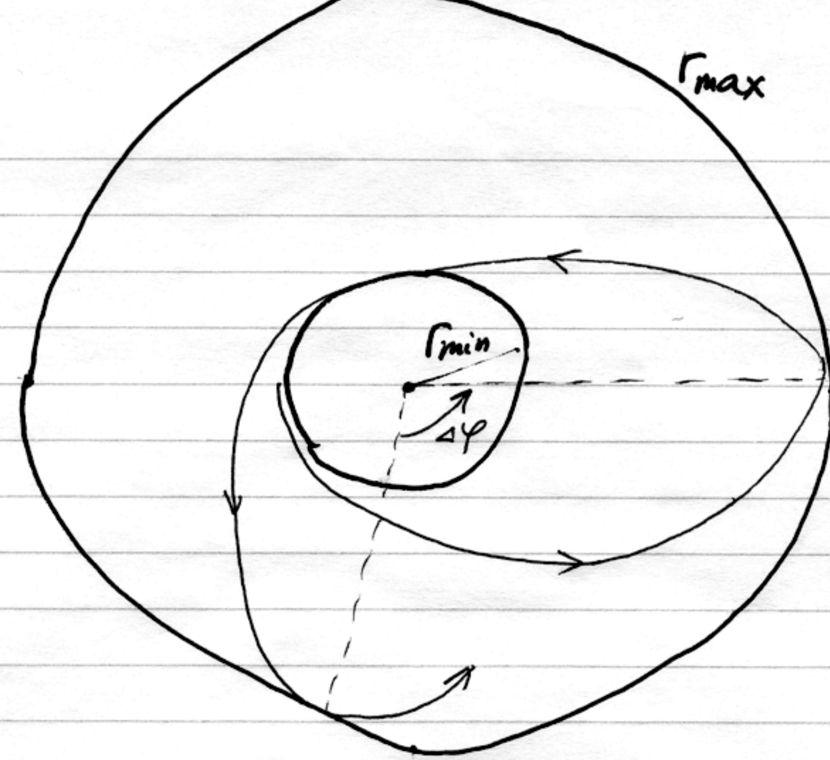
Initial conditions also define plane of the orbit

$$U_{eff}(r) = U(r) + \frac{M^2}{2r^2}$$



Trajectories in the phase-space





Trajectory always stays between r_{min} and r_{max}

The radial period of the motion is equal to

$$T_r = 2\int_{r_{min}}^{r_{max}} \frac{dr}{\sqrt{2(E-U_{eff}(r))}}$$

During one period the apocenter drifts by angle

$$\Delta\varphi = 2\int_{r_{min}}^{r_{max}} \frac{M}{r^2}\frac{dr}{\sqrt{2(E-U_{eff})}}$$

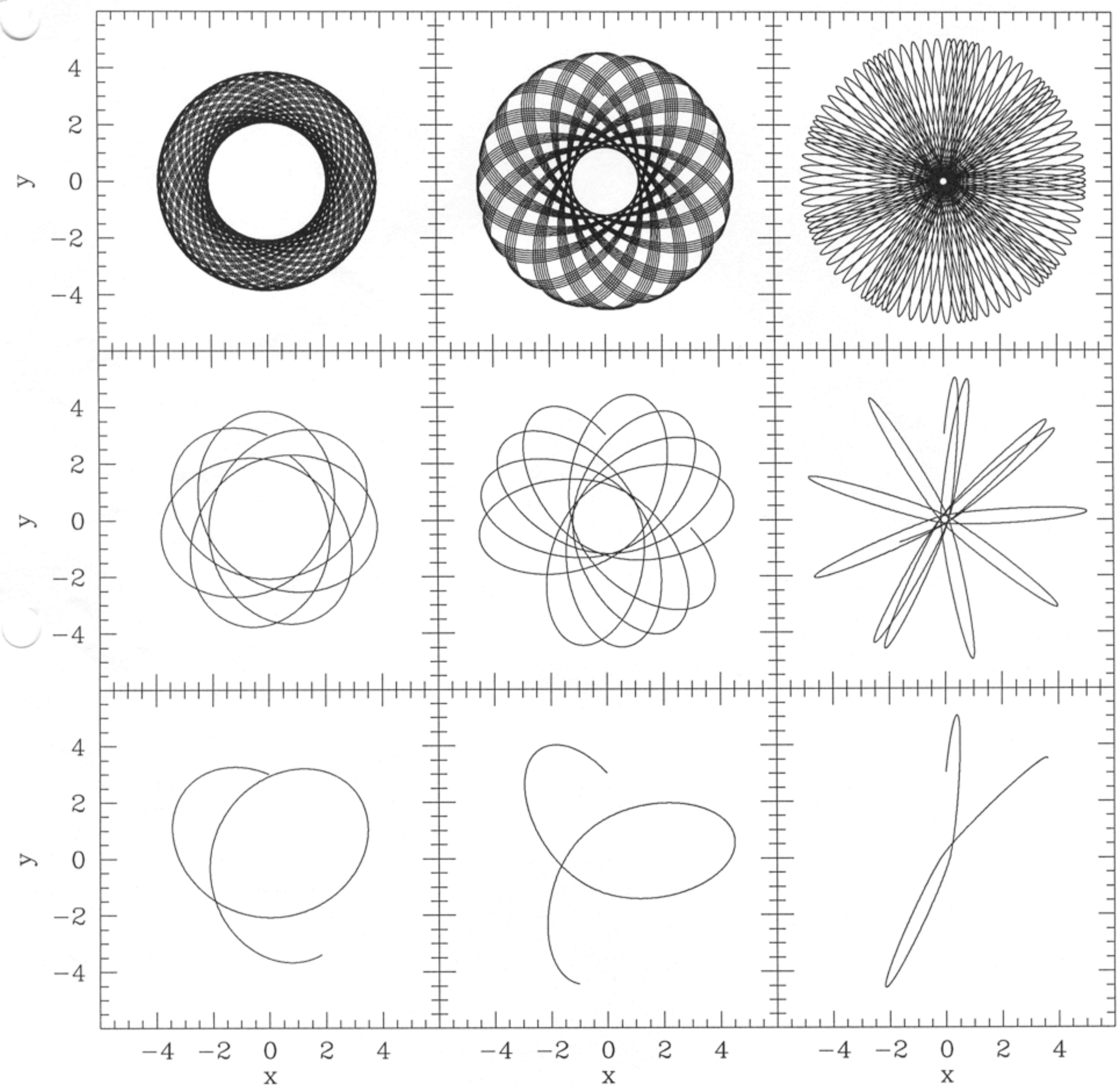
Trajectory is closed if $\Delta\varphi = 2\pi\,\frac{m}{n}$, where m and n are integer numbers

Orbits are closed only in two cases:

$$U(r) = \frac{A}{r} \quad \text{and} \quad U(r) = Ar^2$$

If a trajectory is not close, it covers densily all the space between r_{min} and r_{max}

Orbits gradually fill all allowed space densely: epsilon region around any point will be visited by the trajectory



Examples of motion in central force.

Density: $\rho = \frac{\rho_0}{x(1+x)^2}$ (NFW)

Orbits have the same energy, but different angular momentum.