Bondi accretion

Hydrodynamical equations for a spherical system: Bondi flow

(1) $$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \rho u_r) = 0$$

(2) $$\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} - \frac{u_\theta^2 + u_\varphi^2}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} - \frac{\partial \varphi}{\partial r}$$

For a stationary flow $\frac{\partial ...}{\partial t} = 0$; assume that $u_\theta = u_\varphi = 0$

This gives $$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \rho v) = 0 \quad \| \quad v \equiv u_r$$

$$\Downarrow$$

$$4\pi r^2 \rho v = \dot{M} = const = \text{accreation rate}$$

Gravity is defined by the mass of the central object M:

$$\frac{\partial \varphi}{\partial r} = \frac{GM}{r^2} \Rightarrow \text{(2) is rewritten as}$$

(3) $$v\frac{\partial v}{\partial r} + \frac{1}{\rho}\frac{\partial P}{\partial r} + \frac{GM}{r^2} = 0$$

Polytropic gas : $P = A\rho^\gamma$; $A = const$

$$\frac{\partial P}{\partial r} = \frac{\partial P}{\partial \rho}\frac{\partial \rho}{\partial r} = c_s^2 \frac{\partial \rho}{\partial r} \Rightarrow \frac{1}{\rho}\frac{\partial P}{\partial r} = \frac{c_s^2}{\rho}\frac{\partial \rho}{\partial r}$$

continuity equation gives $$r^2 v \frac{\partial \rho}{\partial r} + \rho \frac{\partial (r^2 v)}{\partial r} = 0 \Rightarrow \frac{\partial \rho}{\rho \partial r} = -\frac{\partial (r^2 v)}{r^2 v \partial r}$$

Thus, eq (3) can be rewritten in the form

$$\frac{1}{2}\frac{\partial v^2}{\partial r} - \frac{c_s^2}{r^2 v}\frac{\partial (r^2 v)}{\partial r} + \frac{GM}{r^2} = 0 \Rightarrow$$

$$\Rightarrow \frac{1}{2}\frac{\partial v^2}{\partial r} - \frac{c_s^2}{r^2 v}\cdot 2r\cdot v - \frac{c_s^2}{v^2}\cdot v\frac{\partial v}{\partial r} + \frac{GM}{r^2} = 0 \Rightarrow \frac{1}{2}\left(1 - \frac{c_s^2}{v^2}\right)\frac{\partial v^2}{\partial r} = -\frac{GM}{r^2}\left[1 - \frac{2c_s^2 r}{GM}\right]$$

analysis of $\frac{1}{2}\left(1-\frac{c_s^2}{v^2}\right)\frac{\partial v^2}{\partial r} = -\frac{GM}{r^2}\left[1-\frac{2c_s^2 r}{GM}\right]$

This equation has 6 types of possible solutions, but not all of them are physical and not all of them correspond to acceation.

— For accreation we expect that at very large radii the gravitation pull from the central mass is negligible. Thus, gas should be at rest at $r \to \infty$: $v(\infty) = 0$

Then the velocity increases as we come closer to the central mass: $\frac{\partial v^2}{\partial r} < 0$

at large r the term in [ ] on the r.h.s. is negative, which means that r.h.s is positive. Because $\frac{\partial v^2}{\partial r} < 0$, in order to have "+" on the l.h.s the velocity must be $\boxed{v^2 < c_s^2}$.

Thus, physical solution for accreation regime is

$\boxed{\text{subsonic at large radii}}$

But at small radii $1-\frac{2c_s^2 r}{GM}$ changes sign and becomes positive. This means that $\underbrace{v^2 > c_s^2}$ =>

$\boxed{\text{at small radii the flow is supersonic}}$

Critical radius: $v = c_s$ $\quad \left\| \; r = r_{crit} = \frac{GM}{2c_s^2(r_{crit})} \; \right\|$

sonic point

$r_{crit} = \frac{GM}{2c_s^2} \simeq 7.5 \cdot 10^{13} \left(\frac{T}{10^4 K}\right)^{-1} \left(\frac{M}{M_\odot}\right)$ cm $\quad$ || much larger than typical size of accreting astronom object

Possible solutions (not all are for accretion)

(1) $v^2 = c_s^2$ at $r = r_{crit}$ $\qquad$ $v^2 \to 0$ as $r \to \infty$

$v^2 < c_s^2$ for $r > r_{crit}$ $\quad$ $v^2 > c_s^2$ at $r < r_{crit}$

(accreation)

(2) $v^2 = c_s^2$ at $r = r_{crit}$ $\qquad$ $v^2 \to 0$ as $r \to 0$

$v^2 > c_s^2$ for $r > r_{crit}$ $\quad$ $v^2 < c_s^2$ at $r < r_{crit}$

(reverse situation to accretion)
stellar wind, Parker solution

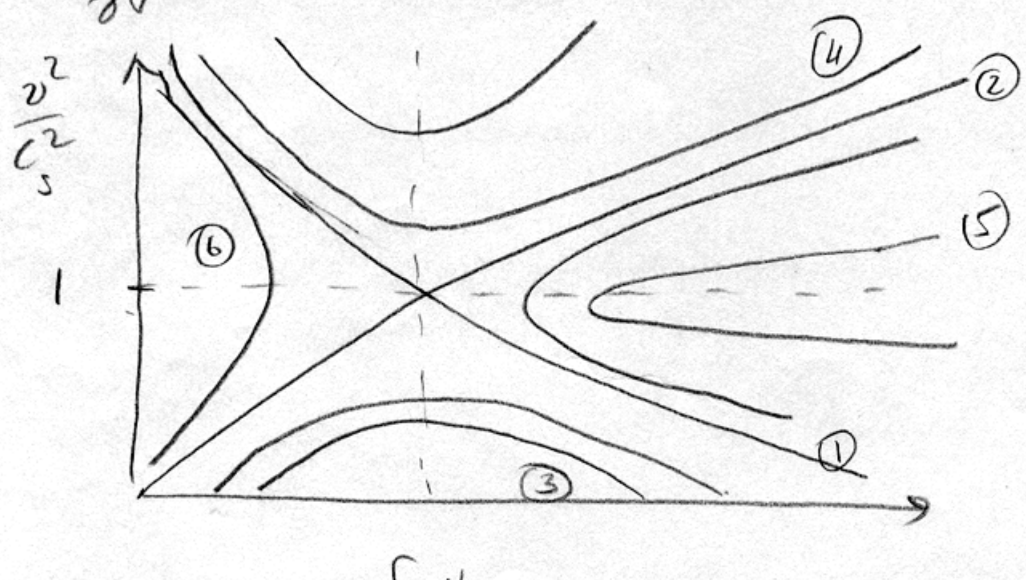
(3) $\dfrac{\partial v^2}{\partial r} = 0$ at $r = r_{crit}$ $\qquad$ $v^2 < c_s^2$ for all radii

(4) $\dfrac{\partial v^2}{\partial r} = 0$ at $r = r_{crit}$ $\qquad$ $v^2 > c_s^2$ for all radii

(5) $\dfrac{\partial v^2}{\partial r} = \infty$ at $r = r_{crit}$ $\qquad$ $r > r_{crit}$ always
$(v^2 = c_s^2)$

(6) $\dfrac{\partial v^2}{\partial r} = \infty$ at $r = r_{crit}$ $\qquad$ $r < r_{crit}$ always

unphysical: two velocities at the same radius



equation $v\frac{\partial v}{\partial r}+\frac{1}{\rho}\frac{\partial P}{\partial r}+\frac{GM}{r^2}=0$

can be integrated once:

$$\frac{v^2}{2}+\frac{c_s^2}{\gamma-1}-\frac{GM}{r}=\text{const}$$

Note: $\int\frac{1}{\rho}\frac{dP}{dr}\cdot dr=\int\frac{dP}{\rho}=$

$=\int\gamma A\rho^{\gamma-1}\frac{d\rho}{\rho}=\gamma A\int\rho^{\gamma-2}d\rho=$

$=\frac{A\gamma}{\gamma-1}\rho^{\gamma-1}=\frac{A\gamma}{\gamma-1}\cdot\frac{\rho^{\gamma}}{\rho}=\frac{\gamma}{\gamma-1}\frac{P}{\rho}=$

$=\frac{c_s^2}{\gamma-1}$

The constant of integration can be found using boundary condition at $r\to\infty$, $v=0$, $c_s=c_s(\infty)$

$$\boxed{\frac{v^2}{2}+\frac{c_s^2}{\gamma-1}-\frac{GM}{r}=\frac{c_s^2(\infty)}{\gamma-1}}$$

Using this equation (Bernoulli) we can relate parameters at ∞ with parameters at the sonic flow:

$$\boxed{r_{crit}=\frac{GM}{c_s^2(\infty)}\frac{(5-3\gamma)}{4}}$$

Note that $r_{crit}=0$ for $\gamma=5/3$

and the sound velocity at the critical (sonic) point is

$$c_s^2(r_{crit})=\frac{2}{5-3\gamma}\cdot c_s^2(\infty)$$

The density at the critical point is

$$\rho(r_{crit})=\rho(\infty)\left[\frac{c_s(r_{crit})}{c_s(\infty)}\right]^{\frac{2}{\gamma-1}}$$

$c_s^2=\gamma\frac{P}{\rho}=\gamma A\rho^{\gamma-1}$

$P=A\rho^{\gamma}$

Accretion rate: $\dot{M}=4\pi r^2\rho v$ can be found:

$$\dot{M}=4\pi r_{crit}^2\rho_{crit}v_{crit}\Rightarrow\dot{M}=\pi G^2M^2\frac{\rho(\infty)}{c_s^3(\infty)}\left[\frac{2}{5-3\gamma}\right]^{\frac{5-3\gamma}{2(\gamma-1)}}$$

OR: $\dot{M}=4\pi\lambda_s\left[\frac{GM}{c_s^2(\infty)}\right]^2\rho_\infty c_s(\infty)$; where $\lambda_s=\left(\frac{1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}\cdot\left(\frac{5-3\gamma}{4}\right)^{-\frac{(5-3\gamma)}{2(\gamma-1)}}$

$\lambda_s=1.120$ for $\gamma=1$

$\lambda_s=0.25$ for $\gamma=5/3$

Asymptotic regimes for transonic solution

$r \gg r_{crit}$ $\quad \rho \simeq \rho_\infty \,, \quad T \simeq T_\infty$

$$v \propto \frac{c_s(\infty) \cdot \lambda_s}{r^2} \left[\frac{GM}{c_s^2(\infty)} \right]^2 \propto \frac{1}{r^2}$$

$r \ll r_{crit}$ $\quad v \simeq \sqrt{\frac{GM}{r}}$ (free-fall speed)

pressure becomes negligible

$$\rho \propto \frac{\lambda_s}{\sqrt{2}} \cdot \rho_\infty \left[\frac{GM}{c_s^2(\infty)} \right]^{3/2} r^{-3/2} \propto r^{-3/2}$$

$\rho r^2 v = \frac{\dot{M}}{4\pi}$

$c_s^2 \propto \rho^{\gamma-1} \propto T$

$$T \propto T_\infty \left[\frac{\lambda_s}{\sqrt{2}} \left[\frac{GM}{c_s^2(\infty)} \right]^{3/2} \right]^{\gamma-1} r^{-\frac{3(\gamma-1)}{2}}$$

Bondi Radius $R_B = \frac{GM}{c_\infty^2}$

accretion Radius $R_A = \frac{2GM}{v_\infty^2}$

Sonic radius $d_s = \frac{5-3\gamma}{4} R_B$

$\Rightarrow$ $v_\infty < c_\infty$ = no shock

= Bondi-type accretion, but density contours are displaced

$\Rightarrow$ $v_\infty \not< c_\infty$ = shock: large γ - bow shock

small γ - tail shock