Boltzman Equation: II

Masses of spherical systems: application of Jeans equations

System to study: stationary spherically symmetric

spherial symmery $\Rightarrow \frac{\partial \ldots}{\partial \varphi} = 0, \; \frac{\partial \ldots}{\partial \theta} = 0$

stationary system $\Rightarrow \frac{\partial \ldots}{\partial t} = 0$

because of the symmetry bulk velocities are equal to zero: $\langle v_\varphi \rangle = \langle v_\theta \rangle = \langle v_r \rangle = 0$

Continuity equation $\frac{\partial n}{\partial t} + \nabla(n\vec{v_s}) = 0$ is satisfied.

The Euler equation gives only one equation:

$$(*) \qquad \frac{\partial}{\partial r}\left(n\overline{v_r^2}\right) + \frac{n}{r}\left[2\overline{v_r^2} - \left(\overline{v_\theta^2} + \overline{v_\varphi^2}\right)\right] = -n\frac{\partial \varphi}{\partial r},$$

where φ is grav. potential and $\overline{v_i^2}$ are velocity dispersions. Because of the spherical symmetry:

$$\langle v_\theta^2 \rangle = \langle v_\varphi^2 \rangle$$

$\langle v_i^2 \rangle \equiv \overline{v_i^2}$

We introduce velocity anisotropy $\beta(r) \equiv 1 - \frac{\langle v_\theta^2 \rangle}{\langle v_r^2 \rangle}$

Eq $(*)$ can be rewritten in the form:

$$\boxed{\frac{1}{n}\frac{\partial}{\partial r}\left(n\overline{v_r^2}\right) + \frac{2\beta \overline{v_r^2}}{r} = -\frac{GM(r)}{r^2}}$$

By rearranging terms:

$$M(r) = -\frac{r\overline{v_r^2}}{G}\left[\frac{\partial \ln n}{\partial \ln r} + \frac{\partial \ln \overline{v_r^2}}{\partial \ln r} + 2\beta\right]$$

This equation can be used to determine mass of spherical systems. It has three quantities, which can be estimated using observational properties of astronomical systems

$n(r) \rightarrow$ reconstructed from surface density of objects $\Sigma(r)$

$\overline{v_r^2} \rightarrow$ from line-of-sight rms velocities

$\beta \rightarrow$ from distribution of l-o-s velocities

__Asymmetric__ __drift__: Rotation of a thin stellar disk

stars rotate around center with velocity $\bar{v}_\varphi$. They also have small chaotic velocities $\overline{v_R^2}$, $\overline{v_\varphi^2}$, and $\overline{v_z^2}$

In cylindrical coordinates one of Jeans' equations can be written as:

$$(*)\quad \frac{\partial}{\partial t}(n\,\bar{v}_R) + \frac{\partial(n\,\overline{v_R^2})}{\partial R} + \frac{\partial(n\,\overline{v_R v_z})}{\partial z} + n\left(\frac{\overline{v_R^2} - \overline{v_\varphi^2}}{R} + \frac{\partial \Phi}{\partial R}\right) = 0$$

Here terms like: $\overline{v_R v_z} = \frac{1}{n}\iiint d^3v\, f \cdot v_R v_z$

Note that there is difference between $\bar{v}_\varphi^2$ and $\overline{v_\varphi^2}$:

$$\sigma_\varphi^2 = \overline{(v_\varphi - \bar{v}_\varphi)^2} = \overline{v_\varphi^2} - \bar{v}_\varphi^2$$

σ_φ^2 = azimuthal velocity dispersion

For a stationary system and close to the galactic plane $\frac{\partial \ldots}{\partial t} = 0$ $\frac{\partial n}{\partial z} \simeq 0$, we have $(*)$

$$(**)\quad \frac{R}{n}\frac{\partial(n\overline{v_R^2})}{\partial R} + R\frac{\partial(\overline{v_R v_z})}{\partial z} + \overline{v_R^2} - \overline{v_\varphi^2} + R\frac{\partial \Phi}{\partial R} = 0$$

Term $\frac{\partial \overline{v_R v_z}}{\partial z}$ is small and we neglect it.

$$R\frac{\partial \Phi}{\partial R} = v_c^2 = (\text{circular velocity})^2$$

__Eq$(**)$ takes form__

$$\boxed{\bar{v}_\varphi^2 = v_c^2 + \overline{v_R^2} - \sigma_\varphi^2 + \frac{R}{n}\frac{\partial(n\overline{v_R^2})}{\partial R}}$$

For solar neighborhood v_R^2 is slightly larger than σ_φ^2, but the difference is not large. The main effect is related with the derivative of "pressure" $\frac{\partial}{\partial R}(n v_R^2)$. Because $n v_R^2$ declines with increasing radius this derivative is negative and

$$\boxed{\overline{v}_\varphi^2 < v_c^2.}$$

Thus, disk rotates slower than $v_c^2 \simeq \frac{GM(R)}{R}$

The larger v_R^2, the strager is the effect.

For an exponential disk with constant hight

$$n(R, z=0) = n_0 e^{-R/r_d}; \quad \sigma_z^2 = \pi G z_0 \Sigma(r)$$

$$\overline{v}_R^2 = v_0^2 e^{-R/r_d}; \quad \sigma_\varphi^2 \simeq 0.45 \overline{v}_R^2$$

$$\boxed{\overline{v}_\varphi^2 = v_c^2 - \sigma_\varphi^2 + \overline{v}_R^2 - 2\frac{R}{r_d}\overline{v}_R^2}$$