Boltzman Equation: I

Boltzman equation

Distribution function: position of each particle in phase-space is characterized by 6 numbers:
$x, y, z, v_x, v_y, v_z \equiv (\vec{x}, \vec{v})$

In 6-dimentional (x, v) space each object is just a point. For that point $\vec{x}$ and $\vec{v}$ are independent variables. Thus, $\vec{x}$ does not depend on $\vec{v}$ and $\vec{v}$ does not depend on $\vec{x}$.

1-point distribution function, or simply DF is defined as a number of of objects in a unit element of the phase-space:

$$\delta N = f(\vec{x}, \vec{v}, t)\,\delta\vec{x}\,\delta\vec{v}; \quad \delta\vec{x} = dx\,dy\,dz$$
$$\delta\vec{v} = dv_x\,dv_y\,dv_z$$

DF $f(\vec{x}, \vec{v}, t)$ defines other (usual) properties of the system. Mean density $n(\vec{x})$ and mean (bulk) velocity $\langle\vec{v}\rangle$ are defined as:

$$n(\vec{x}) = \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} dv_z \cdot f(\vec{x}, \vec{v}, t) \equiv \int d\vec{v}\, f(\vec{x}, \vec{v}, t),$$

$$\langle\vec{v}(\vec{x})\rangle = \frac{1}{n(\vec{x})} \int f(\vec{x}, \vec{v}, t)\,\vec{v}\,d\vec{v}$$

we can define velocity dispersion (analog of pressure in fluids):

$$n(\vec{x})\sigma_{ij}^2 = n(\vec{x})\langle(v_i - \langle v_i\rangle)(v_j - \langle v_j\rangle)\rangle =$$
$$= n(\vec{x})\left(\langle v_i v_j\rangle - \langle v_i\rangle\langle v_j\rangle\right), \text{ where}$$

$$\langle v_i v_j\rangle \equiv \frac{1}{n(\vec{x})} \int v_i v_j f(\vec{x}, \vec{v}, t)\,d\vec{v}$$

1. Derivation of the Boltzman Equation

We describe position of an object in the 6 dimensional phase-space using a vector:

$$\vec{w} = (\vec{x}, \vec{v}). \tag{1}$$

Velocity, with which the object moves in the 6D space, is:

$$\dot{\vec{w}} = (\dot{\vec{x}}, \dot{\vec{v}}) = (\vec{v}, -\vec{\nabla}\phi). \tag{2}$$

The distribution of objects is the phase-space is described by the phase-space density $f(\vec{w}, t)$. Consider a small 6-d cubic volume in the phase-space. The volume of the cube is

$$\Delta V = \Delta x \Delta y \Delta z \Delta v_x \Delta v_y \Delta v_z. \tag{3}$$

The question is how to find the change in mass $\Delta m = \Delta f \Delta V$ of the volume in a Δt time interval. The mass will change because our "fluid" moves through the box: some mass moves in and some mass leaves the volume. Let's count amount of mass, which moves through opposite phases of the box. The box has 6 pairs of phases: 3 in real space and 3 in velocity space. The total change on mass will be the sum of the 6 contributions. The first pair is in the $x-$direction:

$$\Delta m = -\left[(fv_x)|_{x+\Delta x} - (fv_x)|_x\right] \Delta t \Delta S, \tag{4}$$

$$\Delta S = \Delta y \Delta z \Delta v_x \Delta v_y \Delta v_z. \tag{5}$$

Expand in Taylor series:

$$\Delta m = -\frac{\Delta(fv_x)}{\Delta x} \Delta t \Delta x \Delta S \tag{6}$$

Now we do the same for all the phases:

$$\frac{\Delta f}{\Delta t}\Delta V = -\left[\frac{\Delta(fv_x)}{\Delta x} + \frac{\Delta(fv_y)}{\Delta y} + \frac{\Delta(fv_z)}{\Delta z} + \frac{\Delta(f\dot{v_x})}{\Delta v_x} + \frac{\Delta(f\dot{v_y})}{\Delta v_y} + \frac{\Delta(f\dot{v_z})}{\Delta v_z}\right] \Delta V \tag{7}$$

Canceling ΔV and taking the limit of small time and distance intervals, we get:

$$\frac{\partial f}{\partial t} + \mathrm{div}_{6D}\left(f\dot{\vec{w}}\right) = 0 \tag{8}$$

Baltzman equation is derived under condition that no particle can jump in the phase-space. Particles can only smoothly (continiously) move from one place to another.
Thus, the Baltzman equation is just continuity equation of the flow of particles in the phase-space. It can be written in a form:

$$\frac{\partial f}{\partial t} + \underset{6D}{\mathrm{div}} (f \vec{\dot{w}}) = 0$$

Here $\underset{6D}{\mathrm{div}}$ - is the divergence of vector field $f \cdot \vec{\dot{w}}$ in 6D space of $(\vec{x}, \vec{v})$:

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x}(f v_x) + \frac{\partial}{\partial y} \ldots + \frac{\partial}{\partial z} \ldots +$$

$$+ \frac{\partial}{\partial v_x}(f \dot{v}_x) + \frac{\partial}{\partial v_y} \ldots + \frac{\partial}{\partial v_z} \ldots = 0$$

when we unroll the derivatives, half of terms will be equal to zero: $\frac{\partial v_i}{\partial x_i} = 0$ ← because v_i and x_i are independent variables in 6D space

$$\frac{\partial \dot{v}_x}{\partial v_x} = -\frac{\partial}{\partial v_x}\left(\frac{\partial \varphi}{\partial x}\right) = 0 \leftarrow \text{because grav. potential } \varphi \text{ does not depend on } \vec{v}$$

Thus, combining the rest of the terms, we get

$$\frac{\partial f}{\partial t} + (\dot{\vec{w}}\,\nabla_{6D}) f = 0 \Rightarrow \boxed{\frac{\partial f}{\partial t} + (\vec{v}\,\vec{\nabla}_x) f - (\vec{\nabla}\varphi\,\vec{\nabla}_v) f = 0}$$

This is nothing, but a short-cut for

$$\left\| \frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} + v_y \frac{\partial f}{\partial y} + v_z \frac{\partial f}{\partial z} + g_x \frac{\partial f}{\partial v_x} + g_y \frac{\partial f}{\partial v_y} + g_z \frac{\partial f}{\partial v_z} = 0 \right\|$$

Real Boltzman equation has a collisional integral in the right-hand side, which estimates the rate of change of DF due to head-on collisions of atoms in deluted gas. We do not have this term because we assume that there are no collisions between particles.

In plasma and gas physics there is another name associated with this equation: <u>Vlasov</u> <u>equation</u>.

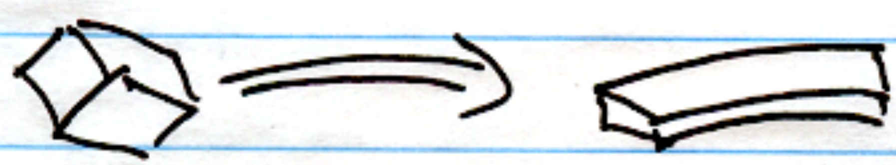
— There are severe limitations on astronomical applications of B.E. because of the way how we defined f and how derived the equation. It implies that there is a small elementary volume of 6D space such that there are (always) many particles inside it AND the distribution function does not change much across the volume. Only in this case we can differentiate $f(\vec{x}, \vec{v})$. In astronomy, this never happens: gradients are typically very large.

- There are two ways to avoid the problem

a) treat f as a probability density: probability to find an object in a small 6D volume element is $dP = f\, d\vec{x}\, d\vec{v}$

b) treat the system as a Hamiltonian flow. In this case we even do not need to have a single real object in space. We just study properties of the Hamiltonian flow.

Liouville theorem: Hamiltonian phase flow preserves volume in the phase-space



This means that we can define the distribution function f at a position of individual star!

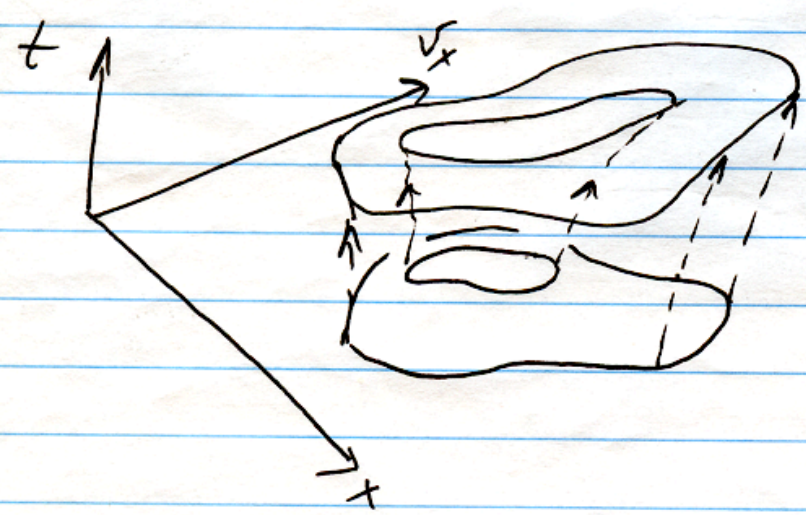
Characteristic equations

The Boltzman equation is the first-order partial differential equation. It is a linear equation. This means that if f_1 and f_2 are two solutions of the equation, then $\alpha f_1 + \beta f_2$ is also a solution, where α and β are arbitrary numbers.

Solution of B.E. can be given in terms of characteristic equations:

$$dt = \frac{dx}{v_x} = \frac{dy}{v_y} = \frac{dz}{v_z} = \frac{dv_x}{\left(-\frac{\partial\varphi}{\partial x}\right)} = \frac{dv_y}{\left(-\frac{\partial\varphi}{\partial y}\right)} = \frac{dv_z}{\left(-\frac{\partial\varphi}{\partial z}\right)}$$

There are 6 differential equations here (e.g. $\frac{dx}{dt} = v_x$, $\frac{dv_x}{dt} = g_x$), which define a curve ("characteristic") in 6D space. The distribution function f is constant along ~~th~~ each curve (but different for different curves). The set of all possible characteristics defines the solution of B.E.



DF is an integral (isolating) of motion.

characteristic equations look like equations of motion of a particle:

$$\frac{d\vec{x}}{dt} = \vec{v}, \quad \frac{d\vec{v}}{dt} = -\frac{\partial \varphi}{\partial \vec{x}}$$

Note that while each particle trajectory is a characteristic, opposite is not true. Characteristics cover the whole space.

Applications of the Boltzman equation in astronomy are often related with its derivative — Jeans equations.

<u>Jeans equations</u> / <u>moment equations</u> / <u>'stellar-hydrodynamics</u>

Boltzman equation: $\frac{\partial f}{\partial t} + (\vec{v}\vec{\nabla})f + (\vec{g}\vec{\nabla}_v)f = 0$

This means that $\frac{\partial f}{\partial t} + \sum_{i=1}^{3}\left\{ v_i \frac{\partial f}{\partial x_i} - \frac{\partial \varphi}{\partial x_i}\frac{\partial f}{\partial v_i}\right\} = 0 \qquad (*)$

Integrate $(*)$ over velocities at given position $\vec{x}$:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dv_x\, dv_y\, dv_z \{B.E\} = 0$$

Consider the three terms in $(*)$ in turn:

I

$$\int d^3v \frac{\partial f}{\partial t} = \frac{\partial}{\partial t}\int d^3v\, f(\vec{x},\vec{v},t) = \frac{\partial}{\partial t} n(\vec{x},t)$$

II $\int d^3v \sum_{i=1}^{3} v_i \frac{\partial f}{\partial x_i} = \sum_{i=1}^{3}\int d^3v \cdot v_i \frac{\partial f}{\partial x_i} = \sum_{i=1}^{3}\frac{\partial}{\partial x_i}\int d^3v\, v_i f(\vec{x},\vec{v},t)$

but $n\langle v_i\rangle = \int v_i f d^3v$, thus II $= \sum_1^3 \frac{\partial}{\partial x_i}[n\langle v_i\rangle] = \mathrm{div}(n\langle\vec{v}\rangle)$

III $\sum_{i=1}^{3} \frac{\partial \varphi}{\partial x_i} \int \frac{\partial f}{\partial v_i} d^3v = \sum_{i=1}^{3} \frac{\partial \varphi}{\partial x_i} \int dv_y \int dv_z \int dv_x \frac{\partial f}{\partial v_i}$

Here each of the three terms has an integral of the form

$$\int_{-\infty}^{\infty} dv_i \frac{\partial f}{\partial v_i} = f \Big|_{v_i=-\infty}^{v_i=+\infty} = 0$$

The integrals are equal to zero because there are no particles moving with infinite velocity

Thus III $= 0$

Combining I, II, and III, we get

$$\boxed{\frac{\partial n}{\partial t} + \mathrm{div}(n \langle \vec{v} \rangle) = 0}$$ Continuity equation

This equation can be considered as equation for density $n(\vec{x})$. It ~~is solved~~ can not be solved because we can not find the mean (= bulk = streaming) velocity $\langle \vec{v} \rangle$ from this equation. Thus, we need to have another equation – equation for $\langle \vec{v} \rangle$.

"Euler equation" is obtained by multiplying B.E. by velocity v_j and by integrating the product over all velocities at given coordinate $\vec{x}$

$$\int d^3v \cdot v_j \{ B.E. \} = 0$$

After some manipulations, we arrive to equation:

$$\frac{\partial \langle v_j \rangle}{\partial t} + \sum_{i=1}^{3} \langle v_i \rangle \frac{\partial}{\partial x_i} \langle v_j \rangle = -\frac{\partial \varphi}{\partial x_j} - \frac{1}{n} \sum_{i=1}^{3} \frac{\partial}{\partial x_i} (n \sigma_{ij}^2)$$

Here tensor σ_{ij}^2 was defined earlier as:

$$\int v_i v_j f \, d^3 v = n \sigma_{ij}^2 + n \langle v_i \rangle \langle v_j \rangle$$

In vector form the euler equation looks a bit better:

$$\boxed{\frac{\partial \langle \vec{v} \rangle}{\partial t} + (\langle \vec{v} \rangle \vec{\nabla}) \langle \vec{v} \rangle = -\vec{\nabla} \varphi - \frac{1}{n} \vec{\nabla} (n \vec{\sigma}^2)}$$

This equation can be considered as equation for $\langle \vec{v} \rangle$.
It can not be solved because we need to know how to find σ_{ij}^2, which is not defined by this equation.
We can proceed further by multiplying BE by higher moments of v's and integrating it over velocities. This produces a hierarchy of equations, which is equivalent to the Boltzman equation.
The hierarchy can be truncated at some stage by assuming some properties of high moments of velocities.
In fluid dynamics we assume that the pressure is isotropic:

$$\sigma_{ij}^2 = \sigma^2 \delta_{ij} \qquad \sigma_{ij}^2 = \begin{pmatrix} \sigma^2 & & 0 \\ & \sigma^2 & \\ 0 & & \sigma^2 \end{pmatrix}$$

Then the last term in the Euler equation is a gradient of pressure. If m is the mean mass of particles and

$$\rho = n \cdot m \text{ is the density,}$$

then

$$P = n \cdot m \sigma^2 .$$

The Euler equation is written in the form $(\vec{v} \equiv \langle \vec{v} \rangle)$

$$\boxed{\frac{\partial \vec{v}}{\partial t} + (\vec{v} \nabla)\vec{v} = -\vec{\nabla}\varphi - \frac{1}{\rho}\vec{\nabla}P}$$